Tight bounds for the vertices of degree k in minimally k-connected graphs

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Abstract

For minimally k-connected graphs on n vertices, Mader proved a tight lower bound for the number $|V_k|$ of vertices of degree k in dependence on n and k. Oxley observed 1981 that in many cases a considerably better bound can be given if $m := |E|$ is used as additional parameter, i.e. in dependence on m, n and k. It was left open to determine whether Oxley's more general bound is best possible.

We show that this is not the case, but give a closely related bound that deviates from a variant of Oxley's long-standing one only for small values of m. We prove that this new bound is best possible. The bound contains Mader's bound as special case.

1 Introduction

Minimally k-connected graphs (i.e. k-connected graphs, for which the deletion of any edge decreases the connectivity) have been in the focus of both structural and extremal graph theory [1, 6] since their early days. For these graphs, the perhaps most heavily investigated parameter is the number $|V_k|$ of vertices of degree k [10].

For $k = 2$, Dirac [2] and Plummer [12] showed that every minimally 2-connected graph contains a vertex of degree 2. In 1969, Halin [4] generalized this result by proving that every minimally k-connected graph contains a vertex of degree k. This proof led to a plethora of further results about the structure of minimally k-connected graphs in general, and $|V_k|$ in particular (see [10] for an extensive survey). In 1979, this eventually culminated in a tight lower bound for $|V_k|$ shown by Mader [9].

While Mader proved that his bound is tight for all $n := |V|$ and k (up to certain parity values), Oxley [11] found, shortly after and inspired by matroids, a different lower bound for $|V_k|$ that uses the parameters m, n and k. Oxley states 1981 that his bound "*frequently sharpens*" Mader's [11]. Since then, classifying the parameters for which Oxley's bound improves Mader's and, even more importantly, finding a lower bound that is generally best possible in dependence on m, n and k, have been open problems.

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We answer both problems by giving a bound that consists of an improvement of Oxley's bound if $m \geq k(kn-1)/(2k-1)$ and of an additional simple bound if $m < k(kn-1)/(2k-1)$. This new bound contains Mader's bound as the special case $m = k(kn-1)/(2k-1)$, and strictly improves even the best-known specialized lower bounds for $k \in \{2,3\}$ given in [3, 11]. As main result, we show that our bound is best possible for all m, $k \geq 2$ and $n > 2k$ (up to certain parity values).

The difficult part of the result is to exhibit suitable infinite families of minimally k-connected graphs in order to prove tightness for both ranges of m mentioned above. The infinite family that we construct for small m may also be of interest in different problem settings, as it consists of minimally k-connected graphs that are "almost k-regular", i.e. such that m is close to a prescribed value slightly above the (minimal possible) value $\lceil kn/2 \rceil$.

After giving the preliminaries, we revisit and generalize the existing lower bounds on $|V_k|$ in Section 3 and give an improvement of Oxley's bound that we will use. We then formulate the new bound and prove its tightness in Section 4.

2 Preliminaries

We consider only finite, simple and undirected graphs. For a graph $G = (V, E)$, let $n_G := |V|$ and $m_G := |E|$ (if G is clear from the context, we omit the subscript). A *k-separator* of a graph is a set of $k \geq 0$ vertices whose deletion leaves a disconnected graph. A graph G is *k-connected* if $n > k$ and G contains no $(k-1)$-separator. A k-connected graph G is *minimally k-connected* if $G - e$ is not k-connected for every edge $e \in E$. Since every non-empty graph is 0-connected according to this definition, there is no minimally 0-connected graph that contains at least one edge. We thus assume $k \geq 1$ throughout this paper.

For a graph G, let $V_k := V_k(G)$ be the set of vertices of degree k and let E_k be the set of edges in G that is induced by V_k. Further, let $F := F(G) := G - V_k$ and let c_F be the number of components of F. If G is minimally k-connected, the following lemmas by Mader ensure that F carries a very special structure.

Lemma 1 ([8, Korollar 1])**.** *For every minimally k-connected graph, F is a forest.*

Lemma 2 ([9, p. 66])**.** *For every minimally k-connected graph, $c_F + |E_k| \geq k$.*

We abbreviate $a \equiv b \pmod{c}$ as $a \equiv_c b$ and write the statement that $a \equiv_c b$ for some $b \in \{b_1, \ldots, b_t\}$ as $a \equiv_c b_1, \ldots, b_t$.

3 Revisiting the old Bounds

Let G be a minimally k-connected graph. We revisit, compare and also generalize the lower bounds for $|V_k|$ that are already known and give small and streamlined proofs for the generalizations. In particular, we show that Oxley's bound can be improved slightly using known methods; this improved variant will be used for our tight bound.

Mader showed that $|V_k| \geq k+1$ and $|V_k| \geq \Delta$ [8, Korollar 2 and Satz 4]. Clearly, the latter bound is at least as good as the former, unless G is k-regular (in which case $|V_k| = n$). However, both bounds are far from being tight.

In his seminal paper [9, Satz 3], Mader eventually proved

$$|V_k| \geq \frac{(k-1)n+2k}{2k-1} \tag{1}$$

and showed that there is a minimally k-connected graph attaining equality in (1) for every k and $n > 2k$ such that $n \equiv_{2k-1} 0, 1, 2, 3, 5, 7, \ldots, 2k-3$. In that sense, Bound (1) is tight for the parameters n and k. We give the following slight generalization of Bound (1), which relates it to Δ.

Theorem 3. *For every minimally k-connected graph,*

$$|V_k| \geq \frac{(k-1)n+2(c_F+|E_k|)+\max\{0, \Delta-(k+1)\}}{2k-1}. \tag{2}$$

Proof. There are exactly $|E(F)| = |V(F)| - c_F = n - |V_k| - c_F$ edges in F. Thus, the number of edges that have exactly one end vertex in F is at least $(k+1)|V(F)| - 2|E(F)| + \max\{0, \Delta-(k+1)\} = (k-1)(n-|V_k|) + 2c_F + \max\{0, \Delta-(k+1)\}$. Counting these edges in dependence on V_k, we obtain $k|V_k| - 2|E_k| \geq (k-1)(n-|V_k|) + 2c_F + \max\{0, \Delta-(k+1)\}$, which gives the claim. □

According to Lemma 2, Bound (2) implies Bound (1). Although Bound (1) is tight for many graphs, it is far from being tight if m is introduced as additional parameter. In fact, we will show in the next section that Bound (1) is only best possible when $m = \frac{k(kn-1)}{2k-1}$.

Using a surprisingly simple proof, Oxley [11, Prop. 2.19][3, Fact 74 in 6.6.12] observed for $k \geq 2$ that $|V_k| \geq \frac{m-n+1}{k-1}$. For the parameters m, n and k, this is the best bound known so far. Since Oxley used $c_F + |E_k| \geq 1$ in his proof, we can apply Lemma 2 and strengthen the bound slightly. In addition, a closer look at the proof of the bound shows that we can actually obtain the following *equality* for V_k.

Theorem 4. *For $k \geq 2$ and every minimally k-connected graph,*

$$|V_k| = \frac{m-n+c_F+|E_k|}{k-1}. \tag{3}$$

In particular, $$|V_k| \geq \left\lceil \frac{m-n+k}{k-1} \right\rceil. \tag{4}$$

Proof. The number of edges that are not in F is $k|V_k| - |E_k|$, as $k|V_k|$ double-counts every edge in E_k. Hence, $m = k|V_k| - |E_k| + |E(F)|$ and $|V_k| = \frac{m+|E_k|-|E(F)|}{k}$. There are exactly $n - |V_k| = |E(F)| + c_F$ vertices of degree greater than k in G, which implies Bound (3). Bound (4) follows from Bound (3) by applying Lemma 2. □

With Bound (3), we have a bound at hand that is always optimal, as long as a minimally k-connected graph with the given parameters exists. Unfortunately, it is not clear at all how to decide whether there is a graph with such a given parameter constellation. We therefore investigate bounds for the rather natural parameters m, n and k.

The given bounds (apart from (3), which is always optimal) relate to each other as follows: For the interesting case $n > 2k$, both bounds (1) and (4) imply $|V_k| \geq \lceil k + \frac{1}{2k-1} \rceil = k + 1$. The bound $|V_k| \geq \Delta$ however is independent of Bounds (1), (2) and (4): Clearly, Δ can be smaller than any of these bounds, as e.g. the k-regular k-connected graphs show. For every sufficiently large wheel graph, $\Delta >$ (1) and $\Delta >$ (4). For every $n > 2k$ and $k > 1$, the graph $K_{k,n-k}$ shows that $\Delta = n - k > \frac{kn-1}{2k-1} =$ (2). The next section will show that Bound (4) is at least as good as (1) if and only if $m \geq k(kn-1)/(2k-1)$ (up to parity issues).

4 A Tight Bound

Harary [5] showed $m \geq \lceil kn/2 \rceil$ for every (minimally) k-connected graph, where $m = (kn+1)/2$ can in fact be attained by such graphs when kn is odd. Mader [7, Satz 2] showed $m \leq kn - \binom{k+1}{2}$ for every minimally k-connected graph, where equality is attained only for the graph K_{k+1} if $k \geq 2$. Thus, every minimally k-connected graph satisfies $\lceil kn/2 \rceil \leq m \leq kn - \binom{k+1}{2}$.

If m is large, our general lower bound for the parameters m, n and k will be (4). If m is small, we use the following lower bound instead, as it outperforms the others in that case. The bound is simple and follows directly from $2m \geq (k+1)(n-|V_k|)+k|V_k|$.

Observation 5. *For every minimally k-connected graph,*

$$|V_k| \geq (k+1)n - 2m. \tag{5}$$

For $k \geq 2$, this gives the general lower bound $|V_k| \geq \max\{(k+1)n - 2m, \lceil (m-n+k)/(k-1) \rceil\}$. In the remaining part of the paper, we show that this bound is tight.

Theorem 6. *For $k \geq 2$ and every minimally k-connected graph G,*

$$|V_k| \geq \begin{cases} (k+1)n - 2m & \text{if } m \leq \frac{k(kn-1)}{2k-1} \\ \lceil (m-n+k)/(k-1) \rceil & \text{if } m \geq \frac{k(kn-1)}{2k-1}. \end{cases} \tag{6}$$

The bound is best possible (even without the ceiling) for every m, $n \geq 3k-2$ and $k \geq 2$ such that

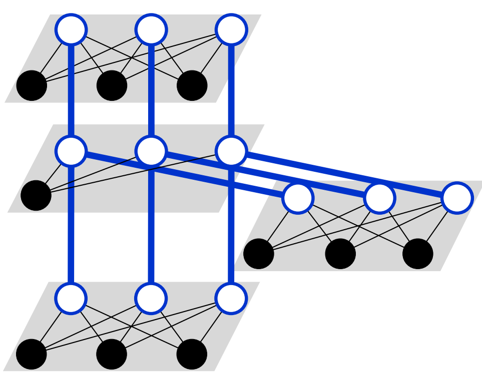
- *$m \equiv_{k(k-1)} k(n-1) - i$ and $0 \leq i \leq 2\lfloor \frac{k}{2} \rfloor$ if $m \leq \frac{k(kn-1)}{2k-1}$, and*
- *$m \equiv_{k-1} k(n-1)$ if $m \geq \frac{k(kn-1)}{2k-1}$.*

Proof. Bound (6) follows directly from the bounds (4) and (5). We prove its tightness under the given assumptions.

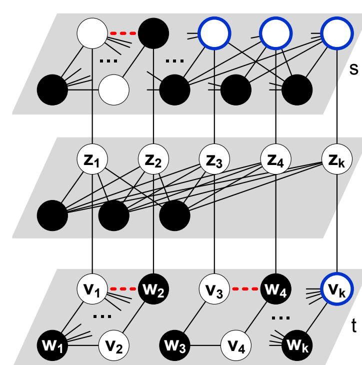
Take k vertex-disjoint copies $T_1, \dots, T_k$ of any tree T with maximum degree at most $k+1$ and $l := |V(T)| \geq 1$. For a vertex $v \in T$, let $v_1, \dots, v_k$ be the vertices in $T_1, \dots, T_k$ that correspond to v; we call this vertex set the *row* of v. Obtain the graph $H_T(k,l)$ from $T_1 \cup \dots \cup T_k$ by adding $k+1-deg_T(v)$ new vertices for each vertex v in T and joining these vertices to each vertex of the row of v by an edge (see Figure 1a). This way, every vertex in a tree copy has degree exactly $k+1$ in $H_T(k,l)$, and $F = T_1 \cup \dots \cup T_k$.

The subgraph that is induced by the vertices of a row and the vertices added to this row is called a *layer*. Every layer of $H_T(k,l)$ is a complete bipartite graph. The graph $H_T(k,l)$ is minimally k-connected, as it can be easily checked that there are k internally vertex-disjoint paths between every vertex pair, and since every edge is either incident to a vertex of degree k or contained in an edge cut of k edges.

We will use $H := H_T(k,l)$ in the construction of tight graph families; to simplify later arguments, we first determine $|V_k(H)|$, n_H and m_H. Since $|V(F(H))| = kl$ and $|E(F(H))| = k(l-1)$ in H, we have $k|V_k(H)| = (k+1)|V(F(H))| - 2|E(F(H))| = (k-1)kl + 2k$, which implies $|V_k(H)| = (k-1)l + 2$ and $n_H = (2k-1)l + 2$. The equality for n_H shows that the construction is well-defined for every $n_H > 2k$ such that $n_H \equiv_{2k-1} 2$, but not well-defined for any $n_H \leq 2k$, as then $l < 1$. It follows that $m_H = k(kl+1)$, which implies $m_H = \frac{k(kn_H-1)}{2k-1}$. Thus, H lies on the threshold of Bound (6). Since $|V_k(H)| = (k+1)n_H - 2m_H = (m_H - n_H + k)/(k-1)$, H satisfies both cases of Bound (6) with equality.



(a) The graph $H_T(3,4)$, where T is the star graph on $l = 4$ vertices. The thick blue subgraph depicts F.



(b) The graph $H'_T(5,3,3)$, where T is the path on $l = 3$ vertices with end vertices s and t. The dashed red lines depict the $i = 3$ edges that were deleted from $H_T(5,3)$ as part of deleting a 1- and a 2-matching in the layers of s and t in order to obtain $H'_T(5,3,3)$. As $j = 1$, the middle row consists only of vertices in V_k. The thick blue subgraph depicts F, which consists of four isolated vertices.

Figure 1: $H_T(k,l)$ and $H'_T(k,l,i)$.

Consider the case $m \leq \frac{k(kn-1)}{2k-1}$ and let $m \equiv_{k(k-1)} k(n-1) - i$ for any $0 \leq i \leq 2\lfloor \frac{k}{2} \rfloor$.

We construct a minimally k-connected graph satisfying $|V_k| = (k+1)n - 2m$. An i-*matching* is a matching of size i. The high-level idea of the construction is to use a modification of H and delete a carefully chosen i-matching as well as j vertices such that the resulting graph is minimally k-connected.

Let $l := \frac{k(n-1)-i-m}{k(k-1)}$. Since $m \equiv_{k(k-1)} k(n-1)-i$, l is an integer. From $m \leq \frac{k(kn-1)}{2k-1}$ follows $l \geq \frac{n-2}{2k-1} - \frac{i}{k(k-1)}$, which implies $l \geq 1 - \frac{1}{k-1}$ due to $n > 2k$ and $i \leq k$. Hence, $l \geq 1$. Let $j := l(2k-1) - n + 2$; clearly, j is an integer.

We prove that $0 \leq j \leq l$ such that $j = l-1$ implies $i \leq \lfloor \frac{k}{2} \rfloor$, and $j = l$ implies $i = 0$. Since $\frac{k(kn-1)}{2k-1} \equiv_{k(k-1)} k(n-1)$, we have $m \equiv_{k(k-1)} \frac{k(kn-1)}{2k-1} - i$. As $m \leq \frac{k(kn-1)}{2k-1}$, this implies $m \leq \frac{k(kn-1)}{2k-1} - i$. Using this bound in the definition of j gives $j \geq 0$. Basic calculus on m, l and j shows that $l = \frac{n-2+(j-l)}{2(k-1)}$ and $m = k(n-1) - i - k(k-1)l$, which implies $m \leq \frac{kn}{2} - i - \frac{k}{2}(j-l)$. If $j \geq l$, we conclude $j = l$ and $i = 0$, since $m \geq \frac{kn}{2}$ (in addition, n is even in this case, as $n = (2k-1)l - j + 2$). If $j = l-1$, n is odd, since $n = (2k-1)l - j + 2$. Then $m \leq \frac{kn}{2} + \frac{k}{2} - i$ implies $i \leq \lfloor \frac{k}{2} \rfloor$, as $m \geq \lceil \frac{kn}{2} \rceil$.

Let T be the path on l vertices, let s and t be its end vertices and let $v \in \{s,t\}$. Then the layer of v in $H_T(k,l)$ is the graph $K_{k,k}$; let its two color classes be black and white such that the row vertices $v_1, \ldots, v_k$ are white and the non-row vertices $w_1, \ldots, w_k$ are black. For any row vertex v_i, let z_i be its (unique) neighbor in $H_T(k,l)$ that is not in the layer of v. Let a *swap* of v_i delete the edge $v_i z_i$ and add the edge $w_i z_i$ (this makes w_i the row vertex instead of v_i). In order to describe the construction, we need the following operation of *deleting an x-matching*, $0 \leq x \leq \lfloor k/2 \rfloor$, in the layer of v (see Figure 1b): For every $1 \leq z \leq x$, perform a swap on the (white) row vertex v_{2z} and delete the edge $v_{2z-1} w_{2z}$. This way, the graph obtained has exactly x edges less, and these edges form an x-matching in $H_T(k,l)$. Since every edge in the x-matching decreases the degree of two vertices of degree $k+1$ by one and does not increase any degree, deleting an x-matching decreases $|V_k|$ by $2x$.

Let $i_t := \min\{i, \lfloor k/2 \rfloor\}$ and $i_s := \max\{0, i - \lfloor k/2 \rfloor\}$; thus, $i_s + i_t = i$. Obtain the graph $H' := H'_T(k,l,i)$ from $H_T(k,l)$ by deleting an i_t-matching in the layer of t, an i_s-matching in the layer of s, and one vertex of degree k from each of j layers that are chosen according to the following preference list on their corresponding vertices in T: inner vertices of T, s, t. This construction is well-defined, because we previously showed $l \geq 1$ (which is needed for the construction of $H_T(k,l)$) and $0 \leq j \leq l$ such that $j = l-1$ implies $i \leq \lfloor \frac{k}{2} \rfloor$ (hence, $i_s = 0$), and $j = l$ implies $i = 0$ (hence, $i_t = i_s = 0$).

By applying Menger's theorem, H' is k-connected: The desired k internally vertex-disjoint paths between all vertex pairs $\{u,v\}$ can be obtained from the ones in $H_T(k,l)$ as follows. First, assume that u and v are in one layer of H'. Then we can substitute every edge of the deleted x-matching with either a path through exactly two vertices in the layer of s or t, or with a path of length 3 in the same layer, such that all substituted paths are pairwise internally vertex-disjoint (see the fully drawn edges in the layer of t in Figure 1b). Otherwise, u and v are in different layers. Then, in all cases, there are vertex-disjoint fans from u to the row of its layer and from v to the row of its layer, and connecting these gives the desired k paths.

Clearly, every edge is incident to a vertex of degree k or contained in an edge cut that consists of k tree-edges; hence, H' is minimally k-connected. Counting edges and vertices of H' in the same way as done for H, we obtain $|V_k(H')| = (k-1)(l+j)+2+2i$, $n_{H'} = (2k-1)l + 2 - j$ and $m_{H'} = k(kl+1-j) - i$. Thus, expanding j in the equality for $n_{H'}$ shows $n_{H'} = n$, and expanding j and substituting l with $\frac{n-2+j}{2k-1}$ in the equality for $m_{H'}$ shows $m_{H'} = m$. Hence, H' satisfies $|V_k(H')| = (k+1)n - 2m$, as claimed.

Now consider the case $m \geq \frac{k(kn-1)}{2k-1}$ and let $m \equiv_{k-1} k(n-1)$. We construct a minimally k-connected graph satisfying $|V_k| = (m-n+k)/(k-1)$. In particular, this shows that Bound (6) is tight without the ceiling. The high-level idea of the construction is to contract $i < k$ suitably chosen edges in H such that the resulting graph is minimally k-connected, followed by adding sufficiently many new vertices of degree k in order to compensate for the vertex loss.

Since $m \equiv_{k-1} k(n-1)$, $\frac{k(n-1)-m}{k-1}$ is an integer. Let $i \in \{0, \ldots, k-1\}$ such that $\frac{k(n-1)-m}{k-1} + i$ is divisible by k; thus, we have $m \equiv_{k(k-1)} k(n-1) + (k-1)i$. Therefore, $l := \frac{k(n-1)-m+(k-1)i}{k(k-1)}$ is an integer.

We prove that $l \geq 1$ and, if $i \neq 0$, $l \geq 2$. Since G is minimally k-connected, $m \leq kn - \binom{k+1}{2}$, where equality is only attained for $G = K_{k+1}$, as mentioned before. Since $G = K_{k+1}$ contradicts $n > 2k$, we have $m < kn - \binom{k+1}{2}$. From $m < kn - \binom{k+1}{2} \leq k(n-1)$ and $i \geq 0$ follows $l > 0$ and thus $l \geq 1$. For $i \geq \frac{k}{2}$, we have $m < kn - \binom{k+1}{2} \leq k(n-1) + (k-1)i - k(k-1)$, which implies $l \geq 2$. Consider the remaining case $1 \leq i < \frac{k}{2}$. Since $n \geq 3k-2$, we can use a result of Mader (see e.g. [1, Thm. 4.9]), which proves $m \leq kn - k^2$. Because $i \geq 1$, we have $m \leq kn - k^2 < kn - k^2 + (k-1)i = k(n-1) + (k-1)i - k(k-1)$, which shows $l \geq 2$. We conclude for all cases $l \geq 1$ and, if $i \neq 0$, $l \geq 2$.

Let $j := n - 2 + i - (2k-1)l$; this will be the number of vertices that is added to the contracted graph. Clearly, j is an integer and, since $i < (2k-1)l$, we have $j \leq n-3$. We prove that $j \geq i$. If $m = \frac{k(kn-1)}{2k-1}$, $m \equiv_{k(k-1)} k(n-1) + (k-1)i$ implies $(2k-1)(k-1)i \equiv_{k(k-1)} 0$ and, as $2k-1$ and k are co-prime, $i = 0$. Since $m \geq \frac{k(kn-1)}{2k-1}$, $m \geq \frac{k(kn-1)}{2k-1} + (k-1)i$ follows from $m \equiv_{k(k-1)} k(n-1) + (k-1)i$. Inserting this lower bound into the definition of l and using the result in the definition of j gives $j \geq i$. Hence, $0 \leq i \leq j \leq n-3$.

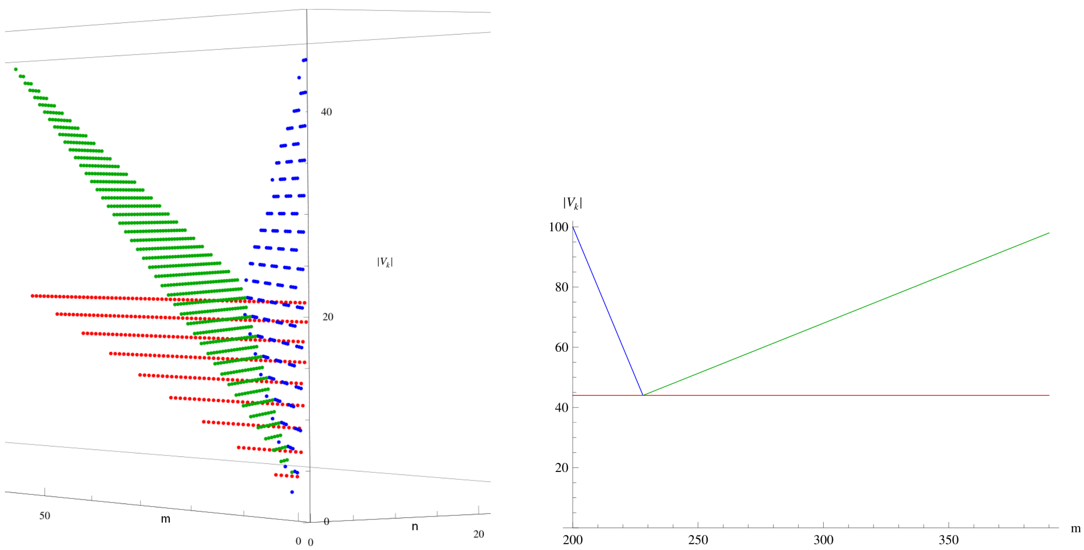
Obtain the graph $H'' := H''_T(k, l, i, j)$ from $H_T(k, l)$ by first adding j new vertices of degree k such that the neighbors of every new vertex are in the same row, and then contracting i edges of (the possibly altered) F that are incident to the k copies of a leaf of T. This construction is well-defined, as we have $l \geq 1$ and, if $i > 0$, the desired $i < k$ edges in F exist due to $l \geq 2$. As before, applying Menger's theorem and reusing internally vertex-disjoint paths from $H_T(k, l)$ for H'' shows that H'' is k-connected. In addition, H'' is minimally k-connected, as every edge e is incident to a vertex of degree k, contained in an edge cut that consists of k tree-edges, or such that $G - e$ contains a $(k-1)$-separator that consists of $k-1$ copies of the leaf chosen in T.

Counting edges and vertices as before, we obtain $|V_k(H'')| = (k-1)l + 2 + j$,

$n_{H''} = (2k-1)l + 2 + j - i$ and $m_{H''} = k(kl+1+j) - i$. Thus, expanding j in the equality for $n_{H''}$ shows $n_{H''} = n$, and expanding j and then l in the equality for $m_{H''}$ shows $m_{H''} = m$. Hence, H'' satisfies $|V_k(H'')| = (m-n+k)/(k-1)$, as claimed. □

In the tightness proof above, the precondition $n \geq 3k-2$ is used only in the case $m \geq \frac{k(kn-1)}{2k-1}$ for the parity values $1 \leq i < \frac{k}{2}$. Hence, for the remaining values $i = 0$ and $\lceil \frac{k}{2} \rceil \leq i \leq k-1$ that satisfy $m \equiv_{k(k-1)} k(n-1) - i$, the weaker precondition $n > 2k$ suffices:

Corollary 7. Bound (6) is best possible (even without the ceiling) for every $k \geq 2$, $n > 2k$ and $m \equiv_{k(k-1)} k(n-1) - i$ such that $\lceil \frac{k}{2} \rceil \leq i \leq 2\lfloor \frac{k}{2} \rfloor$.



(a) A 3D-plot for $k = 3$. A blue ($m \leq k(kn-1)/(2k-1)$) or green ($m \geq k(kn-1)/(2k-1)$) dot at point $(n, m, |V_k|)$ shows the existence of a graph for which Bound (6) is tight. Red dots depict values for which Bound (1) is tight (neglecting m).

(b) A 2D-plot for $k = 4$ and $n = 100$ that shows tight values of Bound (6) (green and blue) and Bound (1) (red) for the relevant ranges of m.

Figure 2: Comparing tight values of Bounds (1) and (6).

Bound (6) implies the best known special-purpose bounds for $k = 2$ and $k = 3$ (see [11, Prop. 2.14+20] and [3, Fact 81]) and improves them for every $m < \lfloor \frac{k(kn-1)}{2k-1} \rfloor$. By comparing Bound (6) with Mader's Bound (1), we obtain immediately that the two bounds match if and only if $m = \frac{k(kn-1)}{2k-1}$. Hence, for the given parities, Mader's bound is only best possible if $m = \frac{k(kn-1)}{2k-1}$; see Figure 2 for a comparison of these two bounds.

While Corollary 7 shows that Bound (6) is tight for $n > 2k$, we leave the problem of determining tight bounds for $n \leq 2k$ as open question. Note that Bound (6) is not tight for $n = 2k$ and $m = \frac{k(kn-1)}{2k-1}$, as every minimally k-connected graph satisfying these constraints has strictly more than $\lceil (m-n+k)/(k-1) \rceil = k+1$ vertices in V_k due to [9, Satz 4].

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