Cubic Plane Graphs on a Given Point Set

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Abstract

Let P be a set of $n \geq 4$ points in the plane that is in general position and such that n is even. We investigate the problem whether there is a (0-, 1- or 2-connected) cubic plane straight-line graph on P. No polynomial-time algorithm is known for this problem. Based on a reduction to the existence of certain diagonals of the boundary cycle of the convex hull of P, we give the first polynomial-time algorithm that checks for 2-connected cubic plane graphs; the algorithm is constructive and runs in time $O(n^3)$. We also show which graph structure can be expected when there is a cubic plane graph on P; e. g., a cubic plane graph on P implies a connected cubic plane graph on P, and a 2-connected cubic plane graph on P implies a 2-connected cubic plane graph on P that contains the boundary cycle of P. We extend the above algorithm to check for arbitrary cubic plane graphs in time $O(n^3)$.

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1 Introduction

Let P be a set of $n \geq 4$ points in the plane that is in *general position*, i. e., that does not contain three points on a line. A *straight-line embedding* of a graph $G = (V, E)$ is an injective function $\pi\colon V \to \mathbb{R}^2$ such that for any two distinct edges ab and cd the straight line segments $\overline{\pi(a)\pi(b)}$ and $\overline{\pi(c)\pi(d)}$ are internally disjoint (i. e., they may only intersect at their endpoints). Let P *admit* a graph $G = (V, E)$ if $|P| = |V|$ and there is a straight-line embedding that maps V to P; we also say that G *is on* P. Thus, P can only admit plane graphs.

We are interested in classifying the point sets P that admit at least one simple plane graph G with a given additional property, e. g., being k-connected, k-edge-connected or k-regular. The graph G is not part of the input: it suffices to find any graph G on P with the desired properties. Using Euler's formula, none of these properties can exist for $k \geq 6$, so we focus on $k \in \{0, \ldots, 5\}$. In addition, there are no k-regular graphs for $k = 1, 3, 5$ when n is odd, since every graph must have an even number of odd vertices. We will assume in these cases that n is even. Since we are dealing only with simple graphs, we can further assume $n > k$ throughout the paper. If not stated otherwise, all graphs are assumed to be simple and plane, but not necessarily connected. All point sets are assumed to be in general position.

For $k \in \{0, 1\}$, it is easy to see that every point set admits a 0-connected 0-regular as well as a connected graph, but a 1-regular graph can only be found when n is even (see Table 1). For $k = 2$, every point set P admits a 2-regular 2-connected (and thus also 2-edge-connected) graph, as there is always a plane cycle on P [4].

k	Necessary and sufficient conditions for a k-connected plane graph	k-edge-connected plane graph	k-regular plane graph	Minimum number of edges in a k-(edge-)connected plane graph on P
0	none	none	none	0
1	none	none	n even	$n-1$
2	none	none	none	n
3	P not in convex position	P not in convex position	? (known for $h \leq \frac{3}{4}n$)	$max(\frac{3}{2}n, n+h-1)$
4	? (known for $h = 3$)	?	?	?
5	?	?	?	?

Table 1: Conditions on P that are both necessary and sufficient for the existence of a k-connected, k-edge-connected and k-regular plane graph, respectively, on a point set P in general position, where $|P| = n > k$.

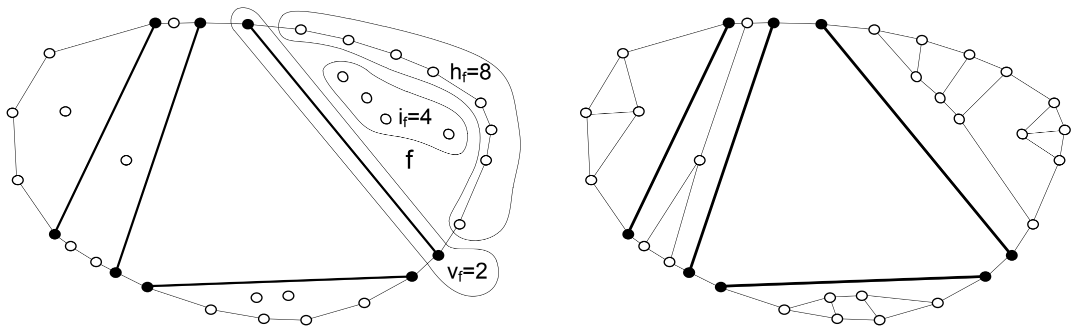
For $k = 3$, Dey et al. [2] give a construction proving that there is a 3-connected graph on P if and only if $n > 3$ and P is not in convex position (the same characterization holds for 3-edge-connected graphs). García et al. [3] investigate how many edges are sufficient to allow a 3-connected graph on P. Let h be the number of points on the convex hull boundary of P. If P is not in convex position, they give a construction of a 3-connected graph on P that has $max(\frac{3}{2}n, n + h - 1)$ edges; the same construction on minimality constraints holds for 3-edge-connected graphs. They also prove that this number is minimal for any 3-connected and for any 3-edge-connected graph on P.

As a corollary, $h \leq \frac{n}{2} + 1$ implies the existence of a 3-regular graph on P. García et al. show in addition that there is a 3-regular graph on P (not necessarily 3-connected, but still 2-connected) when $\frac{n}{2} + 1 \leq h \leq \frac{3}{4}n$ [3, Theorem 4]. While this gives a characterization of the point sets admitting 3-regular graphs for $h \leq \frac{3}{4}n$, the problem remained open for higher values of h. Examples show that the existence of a 3-regular graph is then not any more dependent only on h and n [3]. We give a characterization for all values of h, which leads to the first polynomial time algorithm that computes a 3-regular graph on P if it exists; the running time is $O(n^3)$. We also show that the existence of a 2-connected cubic graph on P implies that there is also a 2-connected cubic graph on P that contains the boundary cycle of the convex hull of P.

If we do not insist on having degree 3 for *every* vertex, the result by Gritzmann et al. [5] that every outerplanar graph is on P can be applied: We can always find a 2-connected outerplanar graph on P that has all except two or three vertices of degree 3. In contrast to $k \leq 3$, very little is known about the case $k \in \{4, 5\}$. There exist point sets that are neither in convex position nor admit 4-connected graphs. For the special case of $h = 3$, Dey et al. [2] could characterize the point sets that admit 4-connected graphs.

Preliminaries. A graph is *cubic* if it is 3-regular. Let P be a set of $n \geq 4$ points in the plane in general position with n even. Let $ch(P)$ denote the boundary cycle of the convex hull of P and let a (combinatorial) edge in $\binom{P}{2}$ be a *diagonal* of P if it joins two non-consecutive points in $ch(P)$. Let H be the set of points in $ch(P)$, $h := |H|$, and let $I = P \setminus H$ be the set of inner points in P, $i := |I|$.

Let D be a set of non-crossing diagonals of P. We call the bounded regions in which $ch(P)$ and D subdivides the plane *faces induced by* D; let $F(D)$ be the set of faces induced by D. For every induced face $f \in F(D)$, let V_f be the set of endpoints of diagonals on the boundary of f, let H_f be the set of points on the boundary of f that are not in V_f and let I_f be the set of points strictly inside f (see Figure 1). Moreover, let $i_f := |I_f|$, $h_f := |H_f|$, $v_f := |V_f|$, $N_f := H_f \dot\cup I_f$ and $n_f := |N_f|$.



(a) The faces induced by a set of non-crossing diagonals. Each diagonal is drawn with a thick edge and black end vertices.

(b) A cubic graph on the point set of Figure 1(a) using the given diagonals.

Figure 1: Induced Faces

For a vertex $v \in V(G)$ and $X \subseteq V(G)$ of a graph G, let $G[X]$ be the subgraph of G that is *vertex-induced* by X. For a simple cubic plane graph G on P and a face $f \in F(D)$, let $G[f] = G[N_f \cup V_f]$.

2 Necessary Conditions for Cubic Graphs

Every simple cubic plane graph G on P contains a (possibly empty) set D of non-crossing diagonals on P. This way, G naturally defines a set of induced faces. Instead of working with concrete cubic graphs, we will consider only their set of diagonals and, for any non-diagonal edge e, the induced face that contains e. It will turn out that it is only important how many non-diagonal edges in an induced face f are incident to a vertex $p \in P$; the precise neighbors of p in f are not crucial. We model this by representing non-diagonal edges with *half-edges*, i. e., we specify for every vertex the number of incident half-edges, but not the neighbors of that vertex.

Definition 1. A *diagonal configuration* $\mathcal{C}$ on P consists of a set $D(C)$ of pairwise non-crossing diagonals of P and a multiset of half-edges on the points of P such that, for every vertex $p \in P$,

- p has degree 3 in C (counting diagonals and half-edges) and
- each half-edge on p is assigned to a face that is induced by $D(C)$ and contains p.

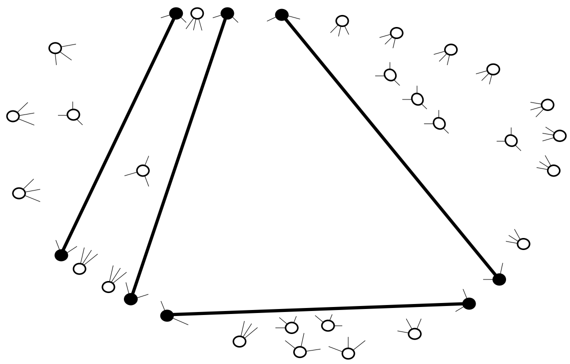


Figure 2: The diagonal configuration of the cubic plane graph in Figure 1(b).

Every simple cubic plane graph G on P determines a unique diagonal configuration C_G by cutting every non-diagonal edge e into two half-edges such that both half-edges are assigned to the induced face that contains e; see Figure 2 for an example. We list necessary conditions on graphs and diagonal configurations to allow cubic graphs on P. García et al. proved the following theorem.

Theorem 2 ([3], implicitly)**.** *Let P be a set of $n \geq 4$ points in general position such that n is even. If $h \leq \frac{3}{4}n$, there is an algorithm with running time $O(n^3)$ that constructs a simple cubic 2-connected plane graph on P that contains $ch(P)$. If $h = n$, P does not admit a simple cubic plane graph.*

We can therefore focus on the case $h > \frac{3}{4}n$; then, every cubic plane graph on P must contain at least one diagonal.

Lemma 3. *Let $h > \frac{3}{4}n$. For every simple cubic (not necessarily connected) plane graph G on P, $D(C_G) \neq \emptyset$. Moreover, $|D(C_G)| \geq 2(h - \frac{3}{4}n) = \frac{h-3i}{2}$.*

Proof. Every vertex v in $H \setminus V(D(C_G))$ has at least one neighbor in I, as v can have at most two neighbors in H and has degree 3. Let s be the number of edges in G that join vertices of H with vertices of I. Then $h - 2|D(C_G)| \leq s$. However, $s \leq 3i$, because every inner vertex can be adjacent to at most 3 vertices in H. It follows that $|D(C_G)| \geq \frac{h-3i}{2} = 2(h - \frac{3}{4}n)$. If $h > \frac{3}{4}n$, then the right-hand side is positive, which concludes the proof. □

To establish further conditions on diagonal configurations for cubic graphs, we classify the vertices in each induced face f. A vertex in V_f can be one of the following types with respect to f (see Figure 3).

Definition 4. Let C be a diagonal configuration on P and let $f \in F(D(C))$. We call a vertex $v \in V_f$ *sated*, *balanced* or *hungry* (*in* f) if its number of incident diagonals and half-edges in f is 1, 2 or 3, respectively.

A vertex in V_f that is not balanced is also called *unbalanced*. Let V_f^0, V_f^+ and V_f^- be the set of *balanced*, *hungry* and *sated* vertices in V_f, respectively. For a sated vertex $v \in V_f^-$, let w be the unique neighbor of v in $ch(P)$ that is contained in f. If w is also sated in f, v is called *matched* (with w) and $\{v, w\}$ is called a *matched vertex pair*. Otherwise, v is *unmatched*. Let V_f^{-m} and V_f^{-u} be the set of *matched* and *unmatched* vertices in V_f, respectively. Let $v_f^0 := |V_f^0|$, $v_f^+ := |V_f^+|$, $v_f^- := |V_f^-|$, $v_f^{-m} := |V_f^{-m}|$ and $v_f^{-u} := |V_f^{-u}|$.

We now assign each induced face f the following integer value $\Delta(f)$ in order to prove necessary conditions on these values for cubic graphs.

(a) $v \in V_f^{-m}$ is sated (and matched) (b) $v \in V_f^{-u}$ is sated (and unmatched) (c) $v \in V_f^0$ is balanced (d) $v \in V_f^+$ is hungry

Figure 3: Diagonal configurations that contain balanced and unbalanced vertices v (with respect to the induced face f).

Definition 5. For a diagonal configuration C on P and each induced face $f \in F(D(C))$, let $\Delta(f) = 3i_f - h_f - v_f^+ - v_f^{-u}$.

It is useful to imagine $h_f + v_f^+ + v_f^{-u}$ as the number of edges between inner and boundary vertices of f that are necessary for any cubic graph on P. Note that an unmatched vertex requires such an edge indirectly, as it forces its non-sated neighbor to require one additional such edge.

Lemma 6. *Let G be a simple cubic plane graph on P. For every face $f \in F(D(C_G))$, $\Delta(f) \geq 0$.*

Proof. Let f be a face in $F(D(C_G))$. We count the number s_f of edges in G that are incident with exactly one vertex in I_f. Each of the v_f^+ hungry vertices in V_f is incident to at least one such edge. There is no diagonal of $H_f \cup V_f$ in G, as otherwise this diagonal would be a diagonal of P due to $H_f \cup V_f \subseteq H$ and contradict f to be an induced face. Therefore, every vertex v in H_f has at most two neighbors in $H_f \cup V_f$ and, hence, at least one neighbor in I_f. Each unmatched vertex $v \in V_f$ has a unique neighbor w in $ch(P) \cap H_f$ such that $vw \notin E(G)$. This forces w to be incident with an additional edge to a vertex in I_f for each such vertex v. It follows that $h_f + v_f^+ + v_f^{-u} \leq s_f$. Since $s_f \leq 3i_f$, $\Delta(f) \geq 0$. □

Let a set of diagonals be *disjoint* if no two of them contain a common end vertex. Note that $D(C_G)$ in Lemma 6 does not have to be disjoint. However, if G contains $ch(P)$, all diagonals are disjoint and consist of balanced vertices only. This gives the following corollary from Lemma 6.

Corollary 7. Let G be a simple cubic plane graph on P that contains $ch(P)$. Then $h_f \leq \frac{3}{4}n_f$ for every induced face $f \in F(D(C_G))$.

We show a necessary parity condition for $\Delta(f)$.

Lemma 8. *Let G be a simple cubic plane graph on P. Then $\Delta(f)$ is even for every induced face $f \in F(D(C_G))$.*

Proof. Consider the graph $G[f]$ and its vertex set $I_f \cup H_f \cup V_f^+ \cup V_f^{-u} \cup V_f^{-m} \cup V_f^0$. By definition, the degree in $G[f]$ of all vertices in V_f^0 is even while the degree of all other vertices in $G[f]$ is odd. As every graph has an even number of odd-degree vertices, $i_f + h_f + v_f^+ + v_f^{-u} + v_f^{-m}$ must be even. However, v_f^{-m} is even, as matched vertices come in pairs. Thus, $i_f + h_f + v_f^+ + v_f^{-u}$ is even and it follows that $3i_f - h_f - v_f^+ - v_f^{-u} = \Delta(f)$ is even. □

3 Constructions

We give sufficient conditions for diagonal configurations to admit cubic graphs. The following result of Tamura and Tamura will be used.

Lemma 9 (Tamura and Tamura [7]). *For points $p_1, p_2, \ldots, p_n$ in general position in the plane and any assignment of degrees from $\{1, 2, \ldots, n-1\}$ to the points such that the sum of degrees is $2n-2$, there is a plane tree with these prescribed degrees on $p_1, p_2, \ldots, p_n$. Moreover, the plane tree can be constructed in time $O(n \log n)$ [1, 3].*

We call an induced face f *empty* if $i_f = h_f = 0$. The following lemmas clarify for which diagonal configurations we can expect cubic plane graphs and, for a special type of configurations, how these graphs can be constructed.

Lemma 10. *Let C be a diagonal configuration without unmatched vertices such that for every non-empty induced face $f \in F(D(C))$, $\Delta(f)$ is even and $0 \leq \Delta(f) < 2i_f$. Then there is a simple cubic plane graph G on P with $D(C_G) = D(C)$. If C has additionally no matched vertices, there is a simple cubic 2-connected plane graph G on P with $D(C_G) = D(C)$ that contains the boundary cycle of P.*

Proof. The proof for the first claim builds on a construction given in [3], but avoids the creation of additional diagonals. Let G' be the graph that consists of $D(C)$ and of all edges in $ch(P)$ that do not join a matched vertex pair. We will construct a cubic graph by adding edges to G' in each induced face $f \in F(D(C))$. As there is no unmatched vertex in $D(C)$, every vertex in $H_f \cup V_f^+$ of an induced face f needs exactly one additional incident edge to the interior of f. Note that the inner vertices I_f in f are not necessarily contained in the convex hull of $H_f \cup V_f^+$ (see, e. g., Figure 4).

In each induced face $f \in F(D(C))$, we augment G' by a collection L of plane trees, each on at least three vertices, such that

(1) the union of all trees is plane,

(2) every vertex v in $I_f \cup H_f \cup V_f^+$ is contained in exactly one tree $T \in L$ and

(3) v has degree 3 in T if $v \in I_f$ and degree 1 in T if $v \in H_f \cup V_f^+$.

In particular, no tree in L contains a boundary edge of f, as it contains at least three vertices. We prove that the second claim can be deduced from the first. As there are neither matched nor unmatched vertices, C has only balanced vertices. Thus, $G' = ch(P) \cup D(C)$ is a cycle with chords. Since every tree in a collection L contains at least two leafs, the constructed graph contains two internally vertex-disjoint paths from every vertex in I_f to distinct boundary vertices of f. Therefore, the constructed graph must be 2-connected if $D(C)$ is disjoint.

It remains to prove the first claim. Let f be any induced face. According to Lemma 8, $\Delta(f) \leq 2i_f - 2$. Let k be the integer such that $\Delta(f) = 2i_f - 2k$, i. e., $1 \leq k \leq i_f$. We first show how to construct a collection of k plane trees $T_1, T_2, \ldots, T_k$ in f that satisfy properties (2) and (3). The first $k-1$ trees are chosen as any collection of vertex-disjoint trees $K_{1,3}$ such that the degree-3 vertex of every $K_{1,3}$ is in I_f.

For convenience, let $z = h_f + v_f^+$. Recall that $\Delta(f) = 3i_f - z$ and that z is the number of distinct boundary vertices of f that need exactly one additional incident edge to some inner vertex in f. Choosing the first $k-1$ trees as described above leaves precisely $i_f - k + 1$ inner vertices of f and $z - 3k + 3$ vertices in $H_f \cup V_f^+$, giving a total of $i_f + z - 4k + 4$ vertices. Thus, the degrees that are still needed for these remaining vertices sum up to $3i_f - 3k + 3 + z - 3k + 3 = 2i_f + 2z - 8k + 6$, which is equal to $2(i_f + z - 4k + 4) - 2$. Therefore, we can apply Lemma 9 to construct a last plane tree T_k with properties (2) and (3).

Now choose L as a collection of plane trees on at least three vertices that satisfies (2) and (3) and for which the sum of all edge lengths is minimal. This collection exists, as we know that $T_1, \ldots, T_k$ exists. We need to show that L satisfies (1). Assume to the contrary that two edges ab and cd that are contained in distinct trees, cross. Using the triangle inequality, we can

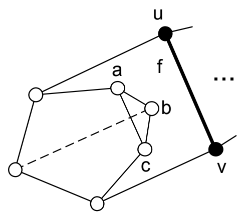


Figure 4: An induced face f with $\Delta(f) = 2i_f = 6$ that does not admit a simple cubic plane graph.

delete ab and cd and either add the edges ac and bd or the edges ad and bc to generate two collections L_1 and L_2 of trees that have a smaller sum of edge lengths. As at most two vertices of $\{a, b, c, d\}$ can be boundary vertices of f, at most one of L_1 and L_2 contains a boundary edge of f. The other collection then preserves properties (2) and (3), which gives a contradiction to L being minimal; thus, L satisfies (1). □

Remark. Intuitively, the precondition $\Delta(f) < 2i_f$ in Lemma 10 avoids that we have to build cycles in the induced face f. If $\Delta(f) \geq 2i_f$, the desired sum of degrees for L would exceed the sum of degrees of every forest in f. The precondition is tight for the statement of Lemma 10, as there are counterexamples even for $\Delta(f) = 2i_f$ ($\Leftrightarrow i_f = h_f + v_f^+$): E. g., in Figure 4, each of the vertices a, b and c has to be adjacent with exactly one of the 3 white boundary vertices for a cubic graph in face f. Thus, a, b and c must form a triangle, which forces every possible edge from b to the boundary to induce a crossing.

For the special case that $\Delta(f) = 0$ for every face f in Lemma 10, each tree will be a $K_{1,3}$. We give an efficient algorithm to construct the trees for this case.

Lemma 11. *Let C be a diagonal configuration without unmatched vertices and $\Delta(f) = 0$ for every induced face $f \in F(D(C))$. There is a $O(n \log n)$ algorithm that constructs a simple cubic plane graph G on P with $D(C_G) = D(C)$ and no edge that joins two points of I.*

Proof. We create a graph G' and, for every induced face $f \in F(D(C))$, a collection L of trees satisfying properties (1)–(3), as shown in the proof of Lemma 10. As $\Delta(f) = 0$, the number of vertices in $H_f \cup V_f^+$ for every face f is exactly $3i_f$ and every such vertex needs exactly one additional incident edge to the interior of f. Thus, L must consist of i_f trees, each of which is a $K_{1,3}$. We take two point sets of equal size: one is $Z := H_f \cup V_f^+$ and the other set Y is generated from I_f by replacing each point $p \in I_f$ with three points that are in a sufficiently small ϵ-neighborhood of p (such an ϵ can be found efficiently).

We need to compute a plane matching between Y and Z. Clearly, it can be assumed that $Y \cup Z$ is in general position. We compute a ham-sandwich cut for Y and Z in time $O(n)$ [6]. Iterating the computation for the subsets of $Y \cup Z$ that are contained in each side of the cut, respectively, will terminate with cells containing exactly one point of Y and one point of Z. Joining the two points for every cell by an edge constructs L, giving a total running time of $O(n \log n)$. □

4 Reduction to Diagonal Configurations

We have shown that a suitable diagonal configuration allows to construct a cubic graph on P. If we can show that every cubic graph on P implies the existence of such a suitable diagonal configuration, we have reduced the problem of computing cubic graphs to diagonal configurations. We will give an efficient algorithm for finding such a diagonal configuration in the next section. Let $h > \frac{3}{4}n$ due to Theorem 2.

Let C_G be the diagonal configuration of a simple cubic plane graph G on P. We will transform C_G to the desired diagonal configuration by iteratively applying four operations. In each step, we maintain a diagonal configuration C, which is initialized with C_G. We stress that the operations are applied in the order of appearance, i.e. an operation is only applied if the operations before are not applicable. Note that all operations transform only diagonal configurations and not graphs (so we forget about the initial graph G).

Operation 1. Let $\Delta(f) > 0$ for an induced face f of $D(C)$ (see, e. g., any graph in Figure 3). Cut a diagonal vw on the boundary of f into two (non-diagonal) half-edges. Both half-edges are assigned to the newly generated induced face z.

Operation 2. Let $v \in V_f^{-u}$ for an induced face f of $D(C)$ (see, e. g., Figures 8(b) and 8(c)). Cut the unique diagonal vw on the boundary of f into two (non-diagonal) half-edges. Both half-edges are assigned to the newly generated induced face z.

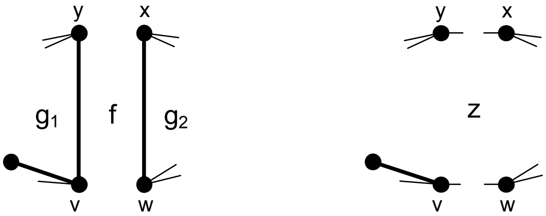
Operation 3. Let f be an induced face of $D(C)$ whose boundary vertices consist of exactly two matched vertex pairs $\{v, w\}$ and $\{x, y\}$ (see Figure 5). Cut the diagonals vy and wx into two (non-diagonal) half-edges each. All half-edges are assigned to the newly generated induced face z.

Operation 4. Let none of the Operations 1–3 be applicable and let f be an induced face of $D(C)$ that contains two matched vertex pairs $\{v, w\}$ and $\{x, y\}$ in this order counterclockwise. We denote the unique neighbors of v and w on the boundary of f by v' and w', respectively (see Figure 6(a)). Note that it is possible that $v' = y$ or $w' = x$ but not both, as Operation 3 is not applicable. Let g_1 and g_2 be the induced faces that are separated from f by vv' and ww', respectively.

Operation 4.1. The quadrangle $\{v, v', w', w\}$ contains no point in I_f.
Cut the diagonals vv' and ww' into two (non-diagonal) half-edges each and add the diagonal $v'w'$ (see Figure 6). Let z and z' be the two new induced faces separated by $v'w'$ such that z' contains v. The two new half-edges at v and w are assigned to z' and all half-edges that were originally in f are assigned to z.

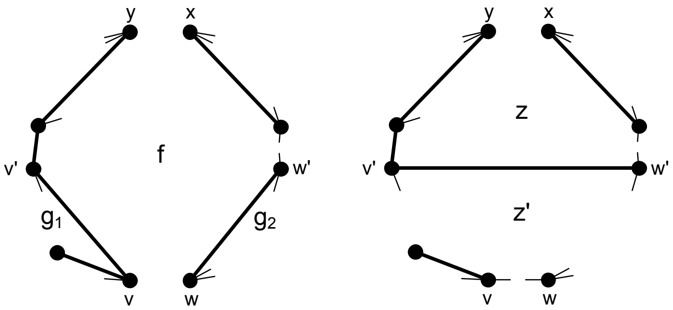
Operation 4.2. The quadrangle $\{v, v', w', w\}$ contains a point in I_f.
We partition $V_f^+ \cup H_f$ into the two subsets X' and Y' such that X' contains the points of $V_f^+ \cup H_f$ to the left of the line $\overline{vy}$ (oriented from v to y) and Y' contains the remaining points of $V_f^+ \cup H_f$. Note that $X' \cup Y'$ does not contain any of the vertices $\{v, w, x, y\}$ and that no unmatched vertex exists due to Operation 2; hence, $\Delta(f)$ is only dependent on i_f, $|X'|$ and $|Y'|$. We set $X = X' \cup \{v'\}$ and $Y = Y' \cup \{w'\}$. Let $x_1, x_2, \ldots, x_s$ with $x_1 = v'$ be the points in X ordered clockwise and let $y_1, y_2, \ldots, y_t$ with $y_1 = w'$ be the points in Y ordered counterclockwise (see Figure 7).
For $i \in \{1, \ldots, s\}$ and $j \in \{1, \ldots, t\}$, let C_{ij} be the diagonal configuration obtained from C by removing the diagonals vv' and ww' and adding the diagonal x_iy_j. Let z and z' be the two new induced faces separated by x_iy_j such that z' contains v. We will replace C by some C_{ij} such that $\Delta(z) = 0$ (the existence of such i and j will be shown in the next lemma). The two new half-edges at v and w are assigned to z'. It remains to assign the half-edges at v', w', x_i and y_j that were originally in f and that are not used for creating the new diagonal x_iy_j: The half-edges at v' and w' are assigned to z'. If $x_i \neq v'$, the half-edges at x_i are distributed to z and z' in the unique way such that x_i is neither sated in z nor in z'; the half-edges at y_j when $y_j \neq w'$ are assigned similarly. Note that Operation 4.2 does not create any new sated vertex in z and z' due to this assignment.

Note that every of the Operations 1–4 decreases the number of diagonals by at least one.



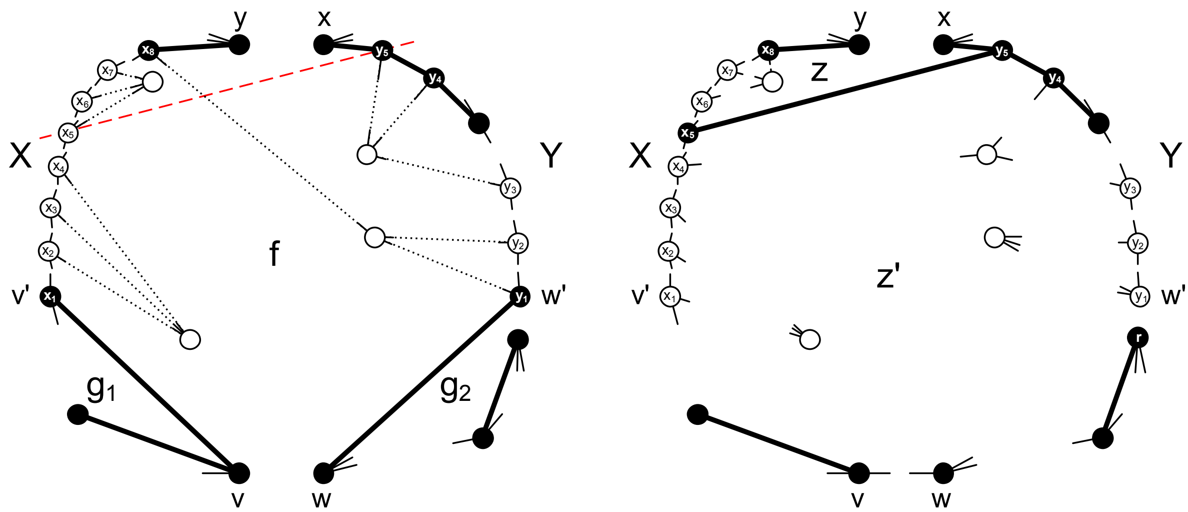
(a) Before Operation 3 (b) After Operation 3.

Figure 5: Operation 3



(a) Before Operation 4.1 (b) After Operation 4.1

Figure 6: Operation 4.1



(a) Before Operation 4.2. The dotted edges depict a possible cubic graph. The dashed edge depicts a choice for i and j such that adding the diagonal $x_i y_j$ creates a face z with $\Delta(z) = 0$.

(b) After Operation 4.2, which deletes the diagonals vv' and ww' and inserts the diagonal $x_5 y_5$. Note that it may be impossible to draw a cubic graph in z', as new unmatched vertices like r may be created. Future applications of Operation 2 will eventually get rid of these unmatched vertices; here, the diagonal at r will be cut.

Figure 7: Operation 4.2

Lemma 12. *In Operation 4.2, $i \in \{1, \dots, s\}$ and $j \in \{1, \dots, t\}$ can be chosen such that $\Delta(z) = 0$ in C_{ij} and such that $i = 1$ or $j = t$.*

Proof. Since Operation 1 is not applicable, $\Delta(f) = 0$ before Operation 4.2. If $i = j = 1$, $\Delta(z) < 0$ in C_{ij}, since $\Delta(f) = 0$ and the quadrilateral $vv'w'w$ contains at least one point of I_f. If $i = s$ and $j = t$, $\Delta(z) \geq 0$ in C_{ij}. Changing C_{ij} to $C_{(i+1)j}$ either increases the value of $\Delta(z)$ by one (if the wedge between the two rays $x_i y_j$ and $x_{i+1} y_j$ contains no point of I_f) or decreases $\Delta(z)$. Similarly, if C_{ij} is changed to $C_{i(j+1)}$, the value of $\Delta(z)$ either increases by one or decreases. It follows that $\Delta(z) = 0$ for at least one of the diagonal configurations $C_{11}, C_{12}, \dots, C_{1t}, C_{2t}, \dots, C_{st}$, giving the claim. ☐

For being able to construct a cubic graph from a diagonal configuration, we need in particular that $\Delta(f) \geq 0$ and $\Delta(f)$ is even for every induced face f. We prove that the above operations preserve these properties of C in every step.

Lemma 13. *Operations 1–4 preserve for every induced face f' that $\Delta(f') \geq 0$ and that $\Delta(f')$ is even.*

Proof. Consider the diagonal configuration C before an Operation 1. As $D(C) \neq \emptyset$ due to $h > \frac{3}{4}n$, at least one diagonal vw exists. Let g be the induced face of $D(C)$ that is separated from f by vw. According to Lemmas 6 and 8, $\Delta(f) \geq 2$ and $\Delta(g) \geq 0$. We check the effects of Operation 1 on $\Delta(z)$ in dependence of the type of v. By symmetry, the same effects hold for w. Figure 8 lists the five possible configurations for vertex types of v in f and g; note that this covers all cases, as v cannot be hungry or sated in both faces, respectively. We illustrate the most involved case $v \in V_f^{-m}$ and $v \in V_g^{+}$ (see Figure 8(e)).

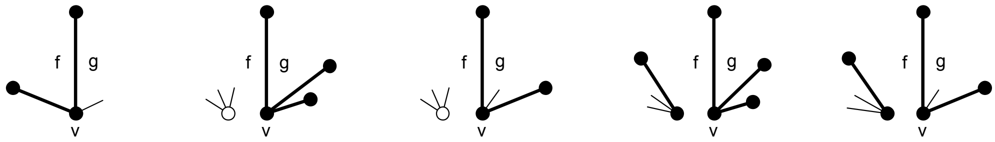
Since $v \in V_f^{-m} \cap V_g^{+}$, the contribution of v to $\Delta(g)$ in C is -1, while the contribution of v to $\Delta(f)$ is 0, as matched vertices do not influence Δ. Applying Operation 1 replaces these two contributions to $\Delta(f)$ and $\Delta(g)$ with one contribution to $\Delta(z)$. The difference between the new and the two old contributions of v is called the *effect* of v on $\Delta(z)$; it reflects the difference of the Δ-values before and after the operation in dependence of v. If v has been incident to only one diagonal of D, v will be a new vertex in H_z (and, thus, in N_z) after performing Operation 1, causing $\Delta(z)$ to decrease by 1. Otherwise, v will be in V_z^{+}, which also decreases $\Delta(z)$ by 1. Additionally, $\Delta(z)$ decreases by another 1 in both cases, since the vertex in f that was formerly matched to v is now unmatched. In total, the effect of v on $\Delta(z)$ is thus $-2 + 1 = -1$. Table 2 lists the effects for the other four cases.

$v \in$	$\Delta(f)$	$v \in$	$\Delta(g)$	$v \in$	$\Delta(z)$	effect on $\Delta(z)$	Fig.
V_f^0	$+0$	V_g^0	$+0$	$H_z \cup V_z^+$	-1	-1	8(a)
V_f^{-u}	-1	V_g^0	$+0$	V_z^0	$+0$	$+1$	8(b)
V_f^{-u}	-1	V_g^+	-1	$H_z \cup V_z^+$	-1	$+1$	8(c)
V_f^{-m}	$+0$	V_g^0	$+0$	V_z^0	$-1^{(\star)}$	-1	8(d)
V_f^{-m}	$+0$	V_g^+	-1	$H_z \cup V_z^+$	$-2^{(\star)}$	-1	8(e)

Table 2: The five possible configurations of a diagonal end vertex v before Operation 1. Columns 2 and 4 depict the contribution of v to the given value. Column 6 depicts the contribution of v to $\Delta(z)$ after Operation 1. For entries marked with $^{(\star)}$, the sated vertex in f that was matched to v becomes unmatched in z.

According to Table 2, $\Delta(z) = \Delta(f) + \Delta(g) + x$ with $x \in \{-2, 0, 2\}$ after applying Operation 1, since exactly two vertices change. This implies $\Delta(z) \geq 0$ and that $\Delta(z)$ is still even.

Consider the diagonal configuration C before an Operation 2. As $v \in V_f^{-u}$, v must be either contained in V_g^0 or in V_g^+ (see Figures 8(b) and 8(c)). In both cases, the effect of v on $\Delta(z)$



(a) v is in V_f^0 and V_g^0 (b) v is in V_f^{-u} and V_g^0 (c) v is in V_f^{-u} and V_g^+ (d) v is in V_f^{-m} and V_g^0 (e) v is in V_f^{-m} and V_g^+

Figure 8: Possible initial configurations for v.

is $+1$, according to Table 2. Thus, after applying Operation 2, $\Delta(z) = \Delta(f) + \Delta(g) + x$ with $x \in \{0, 2\}$ and the claim follows.

Consider the diagonal configuration C before an Operation 3. All four vertices v, w, x and y are matched and therefore either in $V_{g_i}^0$ or in $V_{g_i}^+$ for $i \in \{0, 1\}$ (see Figures 8(d) and 8(e)). Cutting the diagonal vy therefore gives twice an effect of -1 on the new induced face, according to Table 2. Afterwards, w and x are unmatched in this new face. Cutting the diagonal wx then gives twice an effect of $+1$ on $\Delta(z)$. Thus, after applying Operation 3, $\Delta(z) = \Delta(f) + \Delta(g_1) + \Delta(g_2)$ and the claim follows.

Consider the diagonal configuration C before an Operation 4.1. Note that applying Operation 4.1 yields $\Delta(z) = \Delta(f) = 0$, as v and w do not contribute anything to $\Delta(f)$. Note that $\Delta(z')$ differs from $\Delta(f) + \Delta(g_1) + \Delta(g_2) = 0$ only by the effects of the vertices v and w, as $\Delta(z) = 0$. Similarly as for Operation 3, the effects of v and w on $\Delta(z')$ cancel each other. Thus, $\Delta(z') = 0$, which gives the claim.

Consider the diagonal configuration C before an Operation 4.2. Then $\Delta(f) = \Delta(g_1) = \Delta(g_2) = 0$. Since $\Delta(z) = 0$ in C_{ij}, it remains to show that $\Delta(z')$ is even and non-negative. Note that $\Delta(z')$ differs from $\Delta(f) + \Delta(g_1) + \Delta(g_2) = 0$ only by the effects of the vertices $\{v, v', w, w', x_i, y_j\}$, as $\Delta(z) = 0$. The vertices x_i and y_j are both not sated in z and z' and therefore balanced in z and z', as they were formerly contained in $H_f \cup V_f^+$. Thus, the effect of x_i and y_j on $\Delta(z')$ is $+1$ each. The effect of the vertices v and w cancels each other, as shown for Operation 3. The effect of v' if $v' \neq x_i$ and of w' if $w' \neq y_j$ is given by Table 2: Since no vertex is unmatched in C, the effect is -1 per vertex. This amounts to a total effect of $+0$, which gives $\Delta(z') = 0$ and the claim. □

We are now able to state our main structural results.

Theorem 14. *The following statements are equivalent:*

(1) *P admits a simple cubic plane graph G.*

(2) *P admits a simple connected cubic plane graph G'.*

(3) *$h \leq \frac{3}{4}n$ or there is a diagonal configuration C on P such that $\Delta(f) = 0$ for every induced face $f \in F(D(C))$ and no vertex is unmatched.*

Proof. The proof for *(2)* ⇒ *(1)* is immediate. We prove *(1)* ⇒ *(3)*. Assume that $h > \frac{3}{4}n$. Then $D(C_G) \neq \emptyset$ by Lemma 3 and we can iteratively apply (any of the) Operations 1 and 2 on C_G as long as possible until the process terminates with a diagonal configuration C'. Note that the application of each operation decreases the number of diagonals by one; thus, $D(C') \subseteq D(C_G)$. Due to Lemma 13, every induced face $f \in F(D(C'))$ satisfies $\Delta(f) = 0$ and no vertex can be unmatched.

We prove *(3)* ⇒ *(2)*. If $h \leq \frac{3}{4}n$, the proof follows directly from Theorem 2. Assume that $h > \frac{3}{4}n$. Let C' be the diagonal configuration obtained from C by applying Operations 1–4 to C as long as possible. As every operation decreases the number of diagonals by at least one, only finitely many operations are applied. Every face f induced by the diagonals of C' satisfies

$\Delta(f) = 0$ and contains neither an unmatched vertex nor more than one matched vertex pair. Applying Lemma 11 to C' therefore constructs a cubic plane graph on P that is connected. $\square$

Corollary 15. If $h > \frac{3}{4}n$, every simple cubic plane graph on P implies the existence of a simple connected cubic plane graph G on P that contains no unmatched vertex, no edge joining two vertices in I, at most one matched vertex pair for every induced face $f \in F(D(C_G))$ and having $\Delta(f) = 0$ for every such face f.

Theorem 16. *The following statements are equivalent:*

(1) *P admits a simple cubic 2-connected plane graph G.*

(2) *P admits a simple cubic 2-connected plane graph G' such that G' contains $ch(P)$, $C_{G'}$ has no unbalanced vertex and the diagonals $D(C_{G'})$ are disjoint.*

(3) *$h \leq \frac{3}{4}n$ or there is a diagonal configuration C on P such that there are no unbalanced vertices and $h_f = \frac{3}{4}n_f$ for each induced face $f \in F(D(C))$.*

Proof. The proof for *(2)* $\Rightarrow$ *(1)* is immediate. We prove *(3)* $\Rightarrow$ *(2)*. In case that $h \leq \frac{3}{4}n$, Theorem 2 settles the claim. Let $h > \frac{3}{4}n$. Then $D(C)$ must be disjoint, as every vertex v that is end point of two diagonals would contradict that v is balanced in every induced face. With $h_f = \frac{3}{4}n_f$ and $v_f^+ = v_f^- = 0$ for every induced face f, $\Delta(f) = 0$ follows. Applying the construction of Lemma 11 yields the desired graph G' and ensures $ch(P) \subseteq G'$.

It remains to prove *(1)* $\Rightarrow$ *(3)*. We can assume $h > \frac{3}{4}n$. Then G contains at least one diagonal by Lemma 3. As any unbalanced vertex in C_G would imply G to contain a cut vertex, there are no unbalanced vertices in C_G. Iteratively applying Operation 1 on C_G results in a diagonal configuration that satisfies $\Delta(f) = 0$ for every induced face f. As Operation 1 does not introduce new unbalanced vertices, $h_f = \frac{3}{4}n_f$ for every f, which gives the claim. $\square$

The properties we can deduce from not being able to apply Operations 1–4 are valid also for cubic plane graphs that contain the least possible number of diagonals for P.

Corollary 17. If $h > \frac{3}{4}n$, every simple 2-connected cubic plane graph on P that contains the least possible number of diagonals contains $ch(P)$ and no unmatched vertex.

In particular, the number of diagonals in these graphs is completely determined by n and h: Given n and h, the number of diagonals is $2h - \frac{3}{2}n$, which follows from Lemma 3 and the construction of Lemma 11, in which no two vertices in I are joined by an edge. Additionally, $h > \frac{3}{4}n$ implies for every induced face f that every vertex in I_f can be joined to exactly three vertices in H_f. We get the following corollary.

Corollary 18. If $h > \frac{3}{4}n$, every simple cubic 2-connected plane graph on P implies the existence of a simple cubic 2-connected plane graph G on P such that $ch(P) \subseteq G$, $D(C_G)$ is disjoint, $|D(C_G)| = 2h - \frac{3}{2}n$ and there is no edge in G that joins two vertices in I.

One could be tempted to prove that every point set P admitting a connected cubic plane graph also admits a 2-connected cubic plane graph. However, this is not true because of the following counterexample.

Lemma 19. *There is no simple 2-connected cubic plane graph on the point set P in Figure 9.*

Proof. Assume to the contrary there is such a graph G. Since $h > \frac{3}{4}n$, we may assume with Corollary 18 that G contains $ch(P)$ and exactly 3 disjoint diagonals. For every diagonal Z with an end vertex in $\{x, y, z, k, l, m\}$, let $hp(Z)$ be the open halfplane defined by Z that does not contain a. Note that $hp(Z)$ does not contain b either. Assuming there is a diagonal ending at a vertex in $\{x, y, z, k, l, m\}$, we choose one such diagonal Z with a minimal number of points in

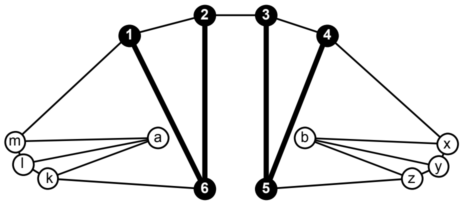


Figure 9: A point set that admits no 2-connected cubic plane graph but a connected cubic plane graph.

$hp(Z)$. As $hp(Z)$ is non-empty, but contains no inner vertex, every vertex in $hp(Z)$ has degree 2, contradicting the cubicness of G. This leaves the six remaining candidates $\{1, \ldots, 6\}$ for an end vertex of a diagonal. No diagonal can join two vertices of $\{1, \ldots, 4\}$, as these vertices form a path in $ch(P)$. Thus, there can be only two disjoint diagonals that have end vertices 5 and 6, respectively, contradicting that there have to be 3 disjoint diagonals. □

With a bit more effort, one can show that the simple cubic plane graph shown in Figure 9 is the only one possible.

5 The Algorithms

In this section we describe two algorithms. We first describe an algorithm for finding a 2-connected cubic plane graph on a given point set P (if it exists). By Theorem 16, it suffices to look for a plane graph G that contains all edges of $ch(P)$ and in which all faces induced by the set of the diagonals of G satisfy $\Delta(f) = 0$. Then we give a similar, very technical algorithm for finding a cubic plane graph (not necessarily connected) on a given point set P (if it exists). In both cases we use a dynamic programming approach. For an ordered pair (a, b) of points, let $R(a, b)$ be the closed halfplane to the right of the line $\overline{ab}$ (oriented from a to b). For a point $x \in H$, let x^+ and x^- be the points of H counterclockwise and clockwise of x, respectively.

Let T be the set of ordered pairs of distinct non-neighboring vertices of $ch(P)$. In the first algorithm, we will compute the following value for each pair $(a, b) \in T$:

Definition 20. Let $d(a, b)$ be the maximum number of a set D of pairwise disjoint diagonals of P such that all these diagonals lie in $R(a, b)$ and all faces induced by them satisfy $\Delta(f) = 0$, with the possible exception of the unique face intersecting the complement of $R(a, b)$.

For $(a, b) \in T$, let $H(a, b)$ be the set of points of H lying in the interior of $R(a, b)$, let $I(a, b)$ be the set of points of I lying in $R(a, b)$, and let $\Delta(a, b) := 3|I(a, b)| - |H(a, b)|$. When computing the numbers $d(a, b)$, we make use of the following observation:

Observation 21. *Let $(a, b) \in T$ and let D be defined as in Definition 20. We distinguish the following two cases.*

(i) D does not contain the diagonal ab.
Let f be the unique face induced by D that contains the segment ab and some points to the right of the segment ab. Then there is a point $c \in H(a, b) \cup \{a\}$ such that the face f contains the pair c, c^+ of adjacent vertices of $ch(P)$. Consequently, every diagonal of D lies entirely in one of the closed halfspaces $R(a, c)$ and $R(c^+, b)$.

(ii) D contains the diagonal ab.
Then all the other diagonals of D lie entirely in $R(a^+, b^-)$.

Algorithm for Finding a 2-Connected Cubic Graph

If n is odd or $n \leq 3$, there is no cubic graph on P; we therefore assume that n is even and $n \geq 4$. If $h \leq \frac{3}{4}n$, we can find a 2-connected cubic plane graph on P in time $O(n^3)$ due to Theorem 2. Let $h > \frac{3}{4}n$. In particular, $h > 3$.

For all pairs $(a, b) \in T$ with $|H(a, b)| \leq 2$, no diagonal in $H(a, b)$ can exist and we set $d(a, b) := 0$. For technical reasons, we also set $d(a, a^+) := d(a, a) := 0$ for each $a \in H$. The values $d(a, b)$ for pairs $(a, b) \in T$ with $|H(a, b)| \geq 3$ are computed in the order of increasing value of $|H(a, b)|$. Thus, we first compute $d(a, b)$ for all pairs $(a, b) \in T$ with $|H(a, b)| = 3$, then for all pairs $(a, b) \in T$ with $|H(a, b)| = 4$, etc.. For each pair $d(a, b)$, we proceed using the following recursion rule.

For a given $(a, b) \in T$ with $|H(a, b)| \geq 3$, let

$$d := \max\{d(a, c) + d(c^+, b) \mid c \in H(a, b) \cup \{a\}\}.$$

Note that no diagonal in $R(a, b)$ is counted twice by taking $d(a, c) + d(c^+, b)$ in the recursion, as $R(a, c)$ and $R(c^+, b)$ are disjoint.

We show how to obtain $d(a, b)$ from d. According to Observation 21(i), the number $d(a, b)$ equals d, unless $d(a, b)$ is witnessed only by sets of diagonals that contain the diagonal ab. In this remaining case, however, $d(a, b)$ equals $d + 1$ by Observation 21(ii), where the $+1$ comes from the additional diagonal ab. Note in this case that, by Definition 20, $\Delta(f) = 0$ also for the induced face f in $R(a, b)$ that contains ab; hence $\Delta(f') = 0$ for all induced faces f'. For this reason and since every diagonal different from ab contains exactly two vertices of $H(a, b)$, we deduce $2d = |H(a, b)| - 3|I(a, b)| = -\Delta(a, b)$.

This gives the following case distinction for computing $d(a, b)$: If $d = d(a^+, b^-)$ and $\Delta(a, b) + 2d = 0$, we can add the diagonal ab to the $d(a^+, b^-)$ diagonals and, thus, set $d(a, b) := d + 1$. Otherwise, we set $d(a, b) := d$.

The computation of $d(a, b)$ takes time $O(n)$ for each pair $(a, b) \in T$. Clearly, all the computation so far can be done in time $O(n^3)$ by dynamic programming. After having computed $d(a, b)$ for all pairs $(a, b) \in T$ in this way, we check if there is a pair $(a, b) \in T$ satisfying

$$2(d(a, b) + d(b, a)) \geq h - 3i.$$

Lemma 22. *If $h > 3$ then P admits a 2-connected cubic plane graph if and only if $2(d(a, b) + d(b, a)) \geq h - 3i$ for some pair $(a, b) \in T$.*

Proof. If P admits a 2-connected cubic plane graph, then by Theorem 16.(3), there is a diagonal configuration C on P with diagonal set $D := D(C)$ such that $\Delta(f) = 0$ for every induced face $f \in F(D)$ and no vertex is unmatched. Choose any diagonal ab of D. By the definition of $d(a, b)$, the number of diagonals of D contained in $R(a, b)$ is at most $d(a, b)$. Similarly, the number of diagonals of D contained in $R(b, a)$ is at most $d(b, a)$. Since each diagonal of D lies in $R(a, b)$ or in $R(b, a)$, we obtain that $d(a, b) + d(b, a) \geq |D|$. Consequently, by Lemma 3, $2(d(a, b) + d(b, a)) \geq |2|D| \geq h - 3i$.

On the other hand, suppose now that $2(d(a, b) + d(b, a)) \geq h - 3i$ for some pair $(a, b) \in T$. Consider the two sets, D_1 and D_2, of diagonals witnessing the values of $d(a, b)$ and $d(b, a)$, respectively. By definition of $d(a, b)$, the set $D_1 \cup D_2$ does not induce any unmatched vertex. If the diagonal ab lies in both D_1 and D_2, each face induced by $D_1 \cup D_2$ satisfies $\Delta(f) = 0$ and we can find a 2-connected cubic plane graph on P due to Lemma 11.

Suppose now that the diagonal ab does not lie in $D_1 \cap D_2$. If it lies neither in D_1 nor in D_2 then $\Delta(f) = 0$ holds for each face induced by the set $D_1 \cup D_2$, with the possible exception of the face containing the segment ab. If the diagonal ab lies exactly in one of the sets D_1 and D_2, then $\Delta(f) = 0$ holds for each face induced by the set $D_1 \cup D_2$, with the possible exception of one face adjacent to the diagonal ab. Due to Lemma 3, removing consequently $\frac{h-3i}{2} - (d(a, b) + d(b, a))$

diagonals at the boundary of the unique face not satisfying $\Delta(f) = 0$ results in a set of diagonals inducing only faces satisfying $\Delta(f) = 0$. Then, due to Lemma 11, there is a 2-connected cubic plane graph on P. □

Thus, if there is no pair $(a, b) \in T$ satisfying $2(d(a, b) + d(b, a)) \geq h - 3i$ then there is no 2-connected cubic plane graph on P. Otherwise, if we find a pair $(a, b) \in T$ satisfying $2(d(a, b) + d(b, a)) \geq h - 3i$, then we use the following recursive procedure which finds diagonals of two sets witnessing the values of $d(a, b)$ and $d(b, a)$, respectively.

Procedure Split(x,y);
If $d(x, y) = d(x^+, y^-) + 1$ and $\Delta(x, y) + 2d(x^+, y^-) = 0$, we add the diagonal xy to D and run Split(x^+, y^-). Otherwise, if $d(x, y) > 0$, we find a $c \in H(x, y) \setminus \{y^-\}$ such that

$$d(x, y) = d(x, c) + d(c^+, y)$$

and then run (for one such c) procedures Split(x, c) and Split(c^+, y).
End of Procedure Split.

Procedure Split follows the way in which the numbers $d(a, b)$ were computed. Therefore, if we first set $D := \emptyset$ and then run procedures Split(a, b) and Split(b, a), we obtain a set D of diagonals such that each face induced by D satisfies $\Delta(f) \geq 0$. This can be done in time $O(n^2)$. We then apply Operation 1 as long as $\Delta(f) > 0$ for some of the induced faces. We obtain a set D' of diagonals such that $\Delta(f) = 0$ holds for each induced face. Then we find pairwise non-intersecting 3-stars in each face covering all inner points in the face according to Lemma 11. These stars in all faces together with the diagonals of D' and with the edges of $ch(P)$ form a 2-connected cubic plane graph on P. The whole algorithm runs in time $O(n^3)$.

Algorithm for Finding any Cubic Graph

If 2-connectivity is not required, our algorithmic approach is similar but more involved than the above algorithm for finding a 2-connected cubic plane graph. The general structure however remains the same: we still maximize the size of a set D of diagonals in $R(a, b)$ for every pair $a, b \in H$ by dynamic programming, where D is the diagonal set of a "near" diagonal configuration C, i.e. of a structure that is a diagonal configuration except possibly for the unique face f intersecting the complement of $R(a, b)$.

According to Corollary 15, we can expect a connected cubic graph that contains no unmatched vertex and no edge joining two points in I and whose induced faces f' satisfy $\Delta(f') = 0$. However, the constraints on C that are necessary to compute such a graph (or a graph sufficiently close to it) have to be considerably more relaxed than the very specific ones used in Definition 20 of the previous algorithm, as here the structure of C is more general: e.g. diagonals may share vertices, matched vertices may exist and we have to specify how many half-edges on a and b leave $R(a, b)$.

To cope with these constraints, we will define $C - ab$ as some special diagonal configuration on $P \cap R(a, b)$ (note that ab is technically not a diagonal of this point set; ab will be fixed later), with the only exception that a and b may have degree less than three in this configuration, namely degree $3 - s$ and $3 - t$ for $s, t \in \{0, 1, 2, 3\}$. Hence, s and t reflect exactly the number of *half-edges* on a and b that leave $R(a, b)$.

The maximizing function of the dynamic program is then extended with the two parameters $s, t \in \{0, 1, 2, 3\}$ and the additional parameter $j \in \{0, 1\}$, which is 1 if the diagonal ab is contained in C and 0 otherwise. In summary, we maximize the following function.

Definition 23. Let $z(a, b, s, t, j)$ be the maximum size of a set D of pairwise non-crossing diagonals of P such that all these diagonals lie in $R(a, b)$ and the following conditions are satisfied:

(i) D contains the diagonal ab if and only if $j = 1$.

(ii) $D - ab$ is the diagonal set of a *diagonal configuration* C on $P \cap R(a,b)$, with the only exception that the degrees of a and b in this configuration are $3 - s$ and $3 - t$ (instead of 3, as in Definition 1).

(iii) Every face f' that is induced by D satisfies $\Delta(f') = 0$, with the possible exception of the unique face f intersecting the complement of $R(a,b)$.

If no such set D exists, $z(a,b,s,t,j)$ is *undefined*. At the beginning of our dynamic program, there are only the following two defined values for $z(a,a^+,s,t,j)$ with $a \in H$. If aa^+ is contained in D, then $j = 1$ and both a and b have both exactly two remaining half-edges, i.e., $z(a,a^+,2,2,1) := 1$. If aa^+ is not contained in D, $z(a,a^+,3,3,0) := 0$. All the other values $z(a,a^+,s,t,j)$ are undefined.

When computing the values $z(a,b,s,t,j)$, we make use of the following analogue of Observation 21:

Observation 24. *Let $(a,b) \in T$ and let D be the set of diagonals as defined in Definition 23. Let f be the unique face induced by D that contains the segment ab and some point to the right of the segment ab. Then f contains a point $c \in H(a,b)$. Consequently, every diagonal of $D - ab$ lies entirely in exactly one of the closed halfspaces $R(a,c)$ and $R(c,b)$.*

We now consider the general case of computing a value $z(a,b,s,t,j)$ for $|H(a,b)| > 0$ (this value may turn out to be undefined). Due to Observation 24,

$$z(a,b,s,t,j) = \max\{j + z(a,c,s',t',j') + z(c,b,s'',t'',j'')\},$$

where the maximum is taken over all $c \in H(a,b)$, all $s', s'', t', t'' \in \{0,1,2,3\}$ and all $j', j'' \in \{0,1\}$, for which the following five conditions are satisfied:

(1) $z(a,c,s',t',j')$ and $z(c,b,s'',t'',j'')$ are defined (consistency)

(2) $s + j \le s'$ (degree condition at a),

(3) $t + j \le t''$ (degree condition at b),

(4) $s'' + t' \ge 3$ (degree condition at c),

(5) if $j = 1$, then $3i = h - 2d + (2 - s) + (2 - t)$, where $d := z(a,c,s',t',j') + z(c,b,s'',t'',j'')$ (condition for $\Delta(f) = 0$)

Condition (1) ensures that only defined values are propagated in the dynamic programming approach. Condition (2) (and, analogously, Condition (3)) ensure that the $s + j$ ($t + j$) half-edges on a (on b) that either leave $R(a,b)$ or belong to the diagonal ab are a subset of the s' (t'') half-edges on a (on b) leaving $R(a,c)$ ($R(c,b)$). Note that the inequality cannot be replaced by equality, since we have to allow possible half-edges on a that are fully contained in $R(a,b)$ but not fully contained in $R(a,c)$ (analogously for half-edges on b).

Similarly, there may be half-edges on c that are neither fully contained in $R(a,c)$ nor $R(c,b)$; each of these contributes $+1$ to both numbers s'' and t'. Thus, we have to allow that $s'' + t' > 3$; however, since c has exactly three incident half-edges and since $s'' + t'$ covers each of these edges, $s'' + t'$ cannot be lower than 3, which gives Condition (3).

So far our computation of z is consistent with the Conditions (i) and (ii) of its Definition 23. For consistency with Condition (*iii*), consider the unique face f that is induced by D and contains the segment ab and some point to the right of ab. If $j = 1$, adding the diagonal ab "closes" f such that f does not intersect the complement of $R(a,b)$ anymore; then we have to ensure that $\Delta(f) = 0$. Note that this automatically ensures consistency with Condition (*iii*) for the case $j = 0$ by applying induction along the dynamic program.

Hence, assume that $j = 1$ and that all faces $f' \neq f$ induced by D satisfy $\Delta(f') = 0$. Let i be the number of inner points in $R(a, b)$, let $h := |H(a, b)|$ and let d be the number $z(a, c, s', t', j') + z(c, b, s'', t'', j'')$ of diagonals in $R(a, b)$ different from ab. Every diagonal that is different from ab decreases the total number of half-edges that are incident to a vertex in $H(a, b)$ by two. Hence, in order to ensure $\Delta(f) = 0$ we have to satisfy $3i = h-2d+(3-s-j)+(3-t-j)$ (i.e., Condition (5)), where $3 - s - j$ and $3 - t - j$ are the numbers of half-edges in $R(a, b)$ on a and b that do not lie on the diagonal ab.

This completes the description of the dynamic program. In exactly the same way as in the previous algorithm, we choose a pair $(a, b) \in T$ that maximizes the total number of diagonals to the left and to the right of the segment ab (using the precomputed values z) and reconstruct a graph G (Procedure Split). As before, G satisfies $\Delta(f') = 0$ for all induced faces f' except for at most one face f that is incident to a or b and has $\Delta(f) > 0$. We did not forbid unmatched vertices so far, so G may contain some. However, after applying Operations 1 and 2 as long as possible, we obtain a diagonal configuration having no unmatched vertex and satisfying $\Delta(f) = 0$ for each induced face. Then we can complete the construction of the desired plane cubic graph by applying Lemma 11. Again, the whole algorithm runs in time $O(n^3)$.

6 Final Remarks

The natural open problems left are to extend the structural results of this paper to 4-regular and 5-regular plane graphs and to find polynomial-time algorithms for recognizing the point sets that admit such graphs.

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