A Simple Test on 2-Vertex- and 2-Edge-Connectivity

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Abstract

Testing a graph on 2-vertex- and 2-edge-connectivity are two fundamental algorithmic graph problems. For both problems, different linear-time algorithms with simple implementations are known. Here, an even simpler linear-time algorithm is presented that computes a structure from which both the 2-vertex- and 2-edge-connectivity of a graph can be easily "read off". The algorithm computes all bridges and cut vertices of the input graph in the same time.

1 Introduction

Testing a graph on 2-connectivity (i. e., 2-vertex-connectivity) and on 2-edge-connectivity are fundamental algorithmic graph problems. Tarjan presented the first linear-time algorithms for these problems, respectively [13, 14]. Since then, many linear-time algorithms have been given (e. g., [2, 3, 5, 6, 7, 8, 15, 16, 17]) that compute structures which inherently characterize either the 2- or 2-edge-connectivity of a graph. Examples include *open ear decompositions* [10, 18], *block-cut trees* [9], *bipolar orientations* [2] and *s-t-numberings* [2] (all of which can be used to determine 2-connectivity) and *ear decompositions* [10] (the existence of which determines 2-edge-connectivity).

Most of the mentioned algorithms use a depth-first search-tree (DFS-tree) and compute the so-called *low-point* values, which are defined in terms of a DFS-tree (see [13] for a definition of low-points). This is a concept Tarjan introduced in his first algorithms and that has been applied successfully to many graph problems later on. However, low-points do not always provide the most natural solution: Brandes [2] and Gabow [8] gave considerably simpler algorithms for computing most of the above-mentioned structures (and testing 2-connectivity) by using simple path-generating rules instead of low-points; they call these algorithms *path-based*.

The aim of this paper is a self-contained exposition of an even simpler linear-time algorithm that tests both the 2- and 2-edge-connectivity of a graph. It is suitable for teaching in introductory courses on algorithms. While Tarjan's two algorithms are currently the most popular ones used for teaching (see [8] for a list of 11 text books in which they appear), in my teaching experience, undergraduate students have difficulties with the details of using low-points.

The algorithm presented here uses a very natural path-based approach instead of low-points; similar approaches have been presented by Ramachandran [12] and Tsin [16] in the context of parallel and distributed algorithms, respectively. The approach is related to ear decompositions; in fact, it computes an (open) ear decomposition if the input graph has appropriate connectivity.

Notation. We use standard graph-theoretic terminology from [1]. Let $\delta(G)$ be the minimum degree of a graph G. A *cut vertex* is a vertex in a connected graph that disconnects the graph upon deletion. Similarly, a *bridge* is an edge in a connected graph that disconnects the graph upon deletion. A graph is 2-connected if it is connected and contains at least 3 vertices, but no cut vertex. A graph is 2-edge-connected if it is connected and contains at least 2 vertices, but no bridge. Note that for very small graphs, different definitions of (edge)connectivity are used in literature; here, we chose the common definition that ensures consistency with Menger's Theorem [11]. It is easy to see that every 2-connected graph is 2-edge-connected, as otherwise any bridge in this graph on at least 3 vertices would have an end point that is a cut vertex.

2 Decomposition into Chains

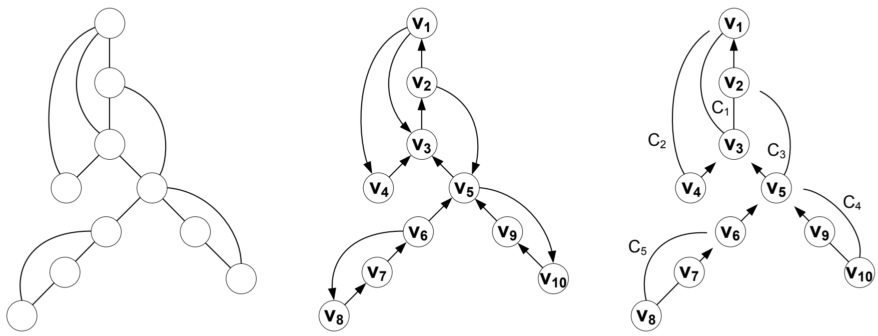
We will decompose the input graph into a set of paths and cycles, each of which will be called a *chain*. Some easy-to-check properties on these chains will then characterize both the 2- and 2-edge-connectivity of the graph. Let $G = (V, E)$ be the input graph and assume for convenience that G is simple and that $|V| \geq 3$. This is not a severe restriction, as self-loops do not influence 2- or 2-edge-connectivity and can therefore be deleted in advance. Similarly, parallel edges do not influence 2-connectivity, but they may influence 2-edge-connectivity, as a bridge does not have parallel edges. However, the 2-edge-connectivity algorithm given in this paper still works for graphs with parallel edges.

We first perform a depth-first search on G. This implicitly checks G on being connected. If G is connected, we get a DFS-tree T that is rooted on a vertex r; otherwise, we stop, as G is neither 2- nor 2-edge-connected. The DFS assigns a *depth-first index* (DFI) to every vertex. We assume that all *tree edges* (i. e., edges in T) are oriented towards r and all *backedges* (i. e., edges that are in G but not in T) are oriented away from r. Thus, every backedge e lies in exactly one *directed cycle* $C(e)$.

Let every vertex be marked as *unvisited*. We now decompose G into *chains* by applying the following procedure for each vertex v in ascending DFI-order: For every backedge e that starts at v, we traverse $C(e)$, beginning with v, and stop at the first vertex that is marked as visited. During such a traversal, every traversed vertex is marked as *visited*. Thus, a traversal stops at the latest at v and forms either a directed path or cycle, beginning with v; we call this path or cycle a *chain* and identify it with the list of vertices and edges in the order in which they were visited. The ith chain found by this procedure is referred to as C_i.

The chain C_1, if exists, is a cycle, as every vertex is unvisited at the beginning (note C_1 does not have to contain r). There are $|E| - |V| + 1$ chains, as every one of the $|E| - |V| + 1$ backedges creates exactly one chain. We call the set $C = \{C_1, \ldots, C_{|E|-|V|+1}\}$ a *chain decomposition*; see Figure 1 for an example.

Clearly, a chain decomposition can be computed in linear time. This almost concludes the algorithmic part; we now state easy-to-check conditions on C



(a) An input graph G.

(b) A DFS-tree of G (depicted with straight-lines) and the edge-orientation it imposes. There are $|E| - |V| + 1 = 5$ backedges.

(c) A chain decomposition $C = \{C_1, \ldots, C_5\}$ of G. The chains C_2 and C_3 are paths; all other chains are cycles. The edge v_6v_5 is not contained in any chain and therefore a bridge. Since $\delta(G) \geq 2$ and $C \setminus C_1$ contains a cycle, G contains a cut vertex (in fact, v_5 and v_6 are cut vertices).

Figure 1: A graph G, its DFS-tree and a chain decomposition of G.

that characterize 2- and 2-edge-connectivity. All proofs will be given in the next section.

Theorem 1. *Let C be a chain decomposition of a simple connected graph G. Then G is 2-edge-connected if and only if the chains in C partition E.*

Theorem 2. *Let C be a chain decomposition of a simple 2-edge-connected graph G. Then G is 2-connected if and only if C_1 is the only cycle in C.*

The properties in Theorems 1 and 2 can be efficiently tested: In order to check whether C partitions E, we mark every edge that is traversed by the chain decomposition. In order to check the property in Theorem 2, we check that C_1 is a cycle and that, for every $i > 1$, the end vertices of C_i are distinct. For pseudo-code, see Algorithm 1.

Algorithm 1 Check(graph G) ▷ G is simple and connected with $|V| \geq 3$

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Compute a DFS-tree T of G
Compute a chain decomposition C; mark every visited edge
if G contains an unvisited edge then
    output "NOT 2-EDGE-CONNECTED"
else if there is a cycle in C different from C_1 then
    output "2-EDGE-CONNECTED BUT NOT 2-CONNECTED"
else
    output "2-CONNECTED"
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We state a variant of Theorem 2, which does not rely on edge-connectivity. Its proof is very similar to the one of Theorem 2.

Theorem 3. *Let C be a chain decomposition of a simple connected graph G. Then G is 2-connected if and only if $\delta(G) \geq 2$ and C_1 is the only cycle in C.*

3 Proofs

It remains to give the proofs of Theorems 1 and 2. For a tree T rooted at r and a vertex x in T, let $T(x)$ be the subtree of T that consists of x and all descendants of x (independent of the edge orientations of T). We will need the following well-known lemma (see, e.g., [4]).

Lemma 4. *An edge is a bridge if and only if it is not contained in any cycle.*

Theorem 1 is immediately implied by the following lemma.

Lemma 5. *Let C be a chain decomposition of a simple connected graph G. An edge e in G is a bridge if and only if e is not contained in any chain in C.*

Proof. Let e be a bridge and assume to the contrary that e is contained in a chain whose first edge (i. e., whose backedge) is b. According to Lemma 4, the bridge e is not contained in any cycle of G. This contradicts the fact that e is contained in the cycle $C(b)$.

Now let e be an edge that is not contained in any chain in C. Let T be the DFS-tree that was used for computing C and let x be the end point of e that is farthest away from the root r of T, in particular $x \neq r$. Then e is a tree-edge, as otherwise e would be contained in a chain. For the same reason, there is no backedge with exactly one end point in $T(x)$. Deleting e therefore disconnects all vertices in $T(x)$ from r. Hence, e is a bridge. □

The following lemma implies Theorem 2, as every 2-edge-connected graph has minimum degree 2.

Lemma 6. *Let C be a chain decomposition of a simple connected graph G with $\delta(G) \geq 2$. A vertex v in G is a cut vertex if and only if v is incident to a bridge or v is the first vertex of a cycle in $C \setminus C_1$.*

Proof. Let v be a cut vertex in G; we may assume that v is not incident to a bridge. Let X and Y be connected components of $G \setminus v$. Then X and Y have to contain at least two neighbors of v in G, respectively. Let X^{+v} and Y^{+v} denote the subgraphs of G that are induced by $X \cup v$ and $Y \cup v$, respectively. Both X^{+v} and Y^{+v} contain a cycle through v, as both X and Y are connected. It follows that C_1 exists; assume w. l. o. g. that $C_1 \notin X^{+v}$. Then there is at least one backedge in X^{+v} that starts at v, since the DFS-tree is rooted in Y^{+v} and X^{+v} contains a cycle through v. When the first such backedge is traversed in the chain decomposition, every vertex in X is still unvisited. The traversal therefore closes a cycle that starts at v and is different from C_1, as $C_1 \notin X^{+v}$.

If v is incident to a bridge, $\delta(G) \geq 2$ implies that v is a cut vertex. Now let v be the first vertex of a cycle $C_i \neq C_1$ in C. If v is the root r of the DFS-tree T that was used for computing C, both cycles C_1 and C_i end at v. Thus, v has at least two children in T and v must be a cut vertex. Otherwise $v \neq r$; let wv be the last edge in C_i. Then no backedge starts at a vertex with smaller DFI than v and ends at a vertex in $T(w)$, as otherwise vw would not be contained in C_i. Thus, deleting v separates r from all vertices in $T(w)$ and v is a cut vertex. □

4 Extensions

We state how some additional structures can be computed from a chain decomposition. Note that Lemmas 5 and 6 can be used to compute all bridges and all cut vertices of G in linear time. Having these, the 2-*connected components* (i. e., the maximal 2-connected subgraphs) of G and the 2-edge-connected components (i. e., the maximal 2-edge-connected subgraphs) of G can be easily obtained: it suffices to cut the DFS-tree T along all cut-vertices or, respectively, all bridges. The former also gives the so-called *block-cut tree* [9] of G, which is a tree representing the dependency of the 2-connected components and cut vertices of G. Similarly, cutting all bridges in T gives a tree that represents the dependency of the 2-edge-connected components and bridges of G.

Additionally, the set of chains C computed by our algorithm is an ear decomposition if G is 2-edge-connected and an open ear decomposition if G is 2-connected. Note that C is not an arbitrary (open) ear decomposition, as it depends on the DFS-tree. The existence of these ear decompositions characterize the 2-(edge-)connectivity of a graph [10, 18]; Brandes [2] gives a simple linear-time transformation that computes a *bipolar orientation* and an *s-t-numbering* from such an open ear decomposition.

References

[1] J. A. Bondy and U. S. R. Murty. *Graph Theory.* Springer, 2008.

[2] U. Brandes. Eager st-Ordering. In *Proceedings of the 10th European Symposium of Algorithms (ESA'02)*, pages 247–256, 2002.

[3] J. Cheriyan and K. Mehlhorn. Algorithms for dense graphs and networks. *Algorithmica*, 15(6):521–549, 1996.

[4] R. Diestel. *Graph Theory.* Springer, fourth edition, 2010.

[5] J. Ebert. st-Ordering the vertices of biconnected graphs. *Computing*, 30:19–33, 1983.

[6] S. Even and R. E. Tarjan. Computing an st-Numbering. *Theor. Comput. Sci.*, 2(3):339–344, 1976.

[7] S. Even and R. E. Tarjan. Corrigendum: Computing an st-Numbering (TCS 2(1976):339-344). *Theor. Comput. Sci.*, 4(1):123, 1977.

[8] H. N. Gabow. Path-based depth-first search for strong and biconnected components. *Inf. Process. Lett.*, 74(3-4):107–114, 2000.

[9] F. Harary and G. Prins. The block-cutpoint-tree of a graph. *Publ. Math. Debrecen*, 13:103–107, 1966.

[10] L. Lovász. Computing ears and branchings in parallel. In *Proceedings of the 26th Annual Symposium on Foundations of Computer Science (FOCS'85)*, pages 464–467, 1985.

[11] K. Menger. Zur allgemeinen Kurventheorie. *Fund. Math.*, 10:96–115, 1927.

[12] V. Ramachandran. Parallel open ear decomposition with applications to graph biconnectivity and triconnectivity. In *Synthesis of Parallel Algorithms*, pages 275–340, 1993.

[13] R. E. Tarjan. Depth-first search and linear graph algorithms. *SIAM Journal on Computing*, 1(2):146–160, 1972.

[14] R. E. Tarjan. A note on finding the bridges of a graph. *Inf. Process. Lett.*, 2(6):160–161, 1974.

[15] R. E. Tarjan. Two streamlined depth-first search algorithms. *Fund. Inf.*, 9:85–94, 1986.

[16] Y. H. Tsin. On finding an ear decomposition of an undirected graph distributively. *Inf. Process. Lett.*, 91:147–153, 2004.

[17] Y. H. Tsin and F. Y. Chin. A general program scheme for finding bridges. *Inf. Process. Lett.*, 17(5):269–272, 1983.

[18] H. Whitney. Non-separable and planar graphs. *Trans. Amer. Math. Soc.*, 34(1):339–362, 1932.