Cubic Plane Graphs on a Given Point Set

Jens M. Schmidt Pavel Valtr

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point set P in the plane in general position; $n := |P| > 3$

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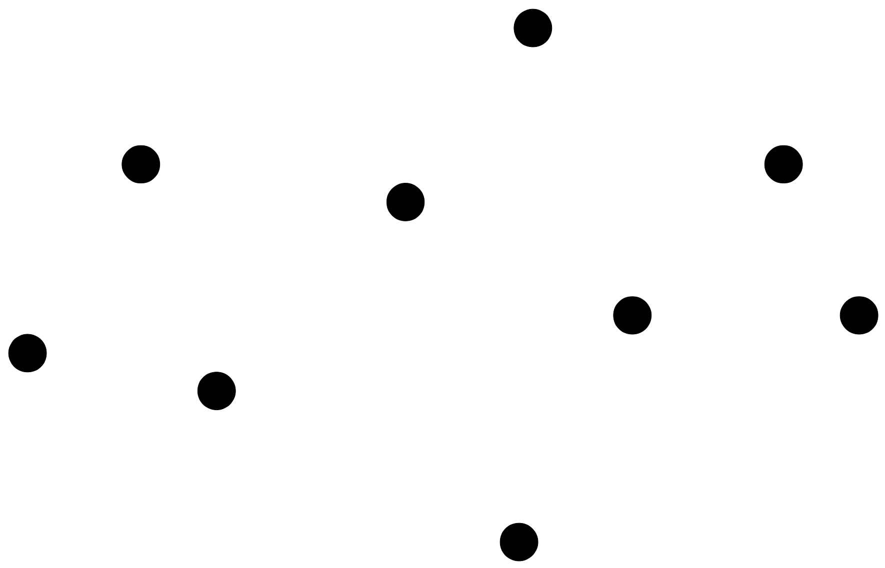

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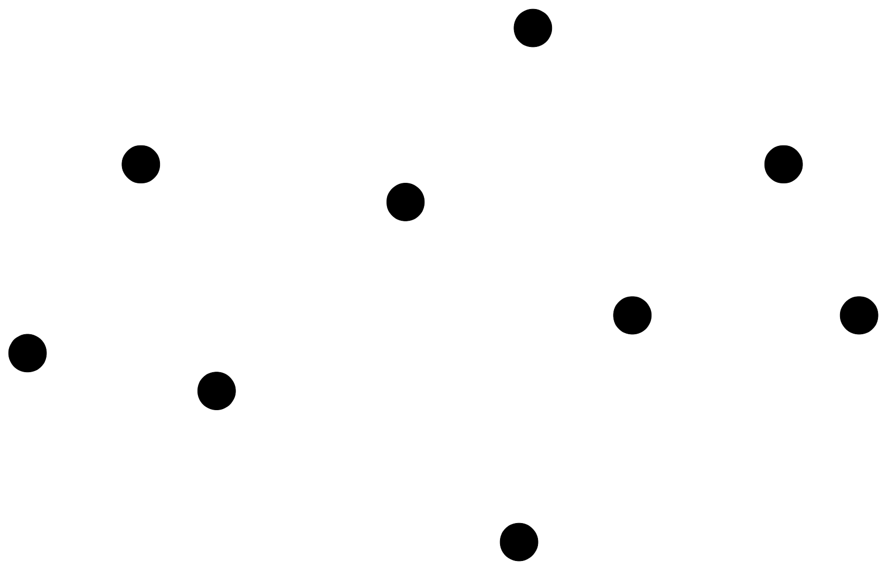

Is there a planar straight-line drawing of a connected graph on P?

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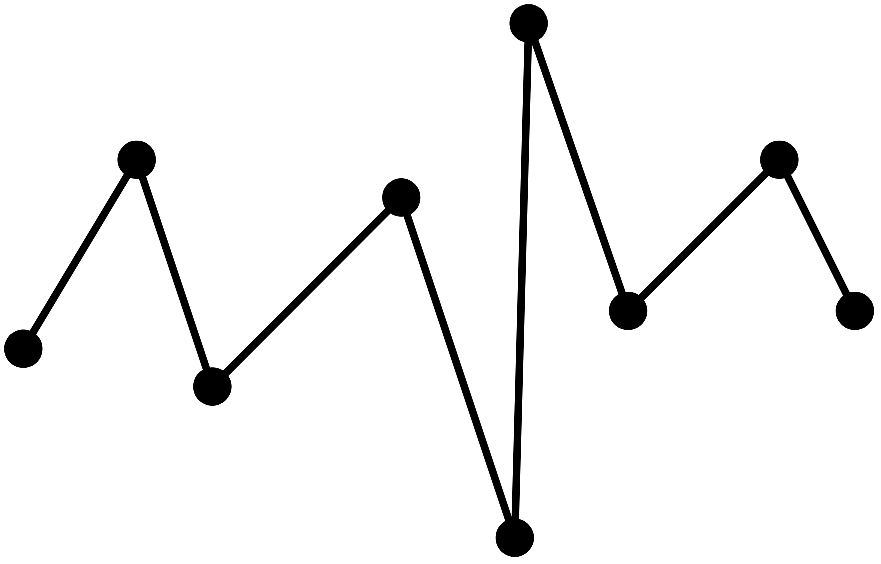

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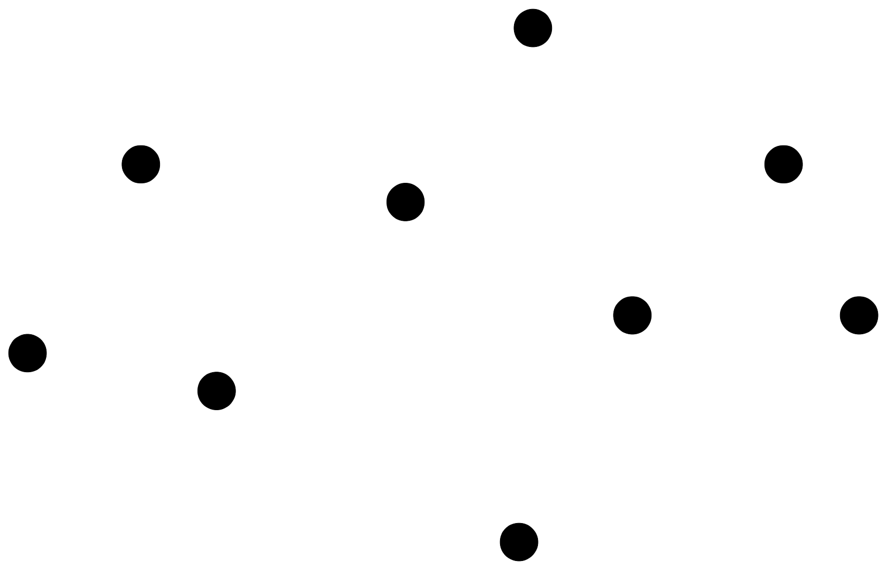

Is there a planar straight-line drawing of a 2-connected graph on P?

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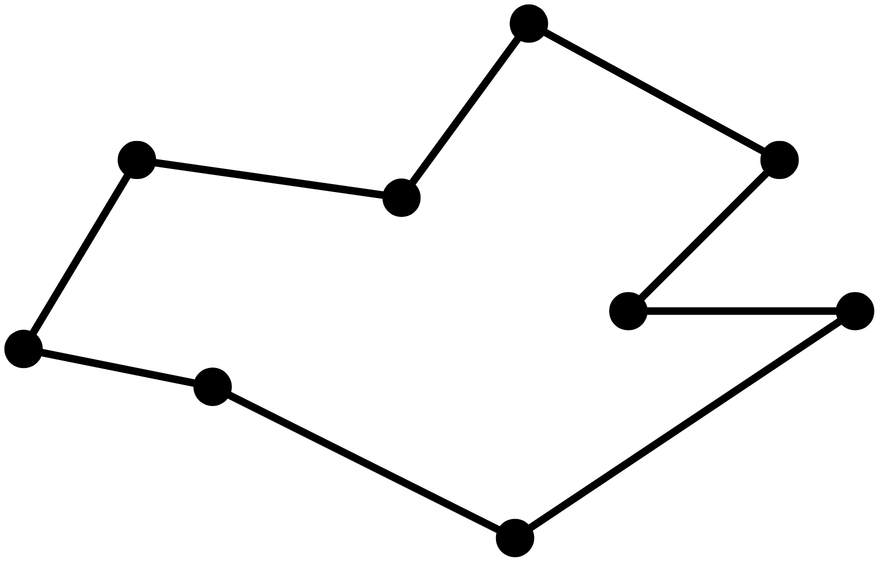

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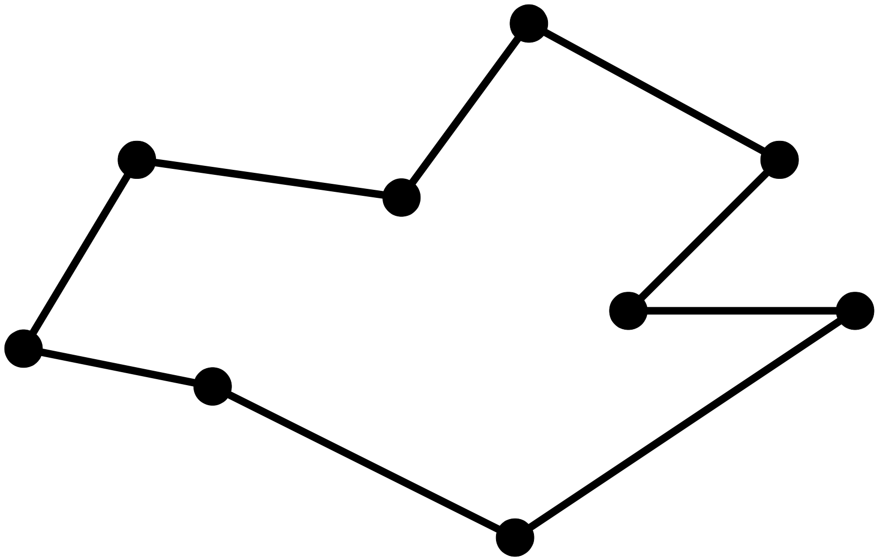

Is there a planar straight-line drawing of a 2-connected graph on P?

2-regular

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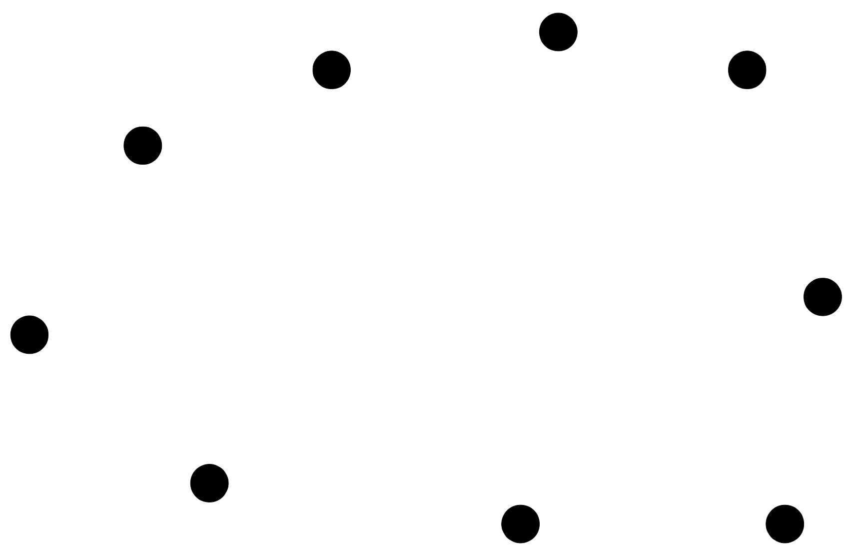
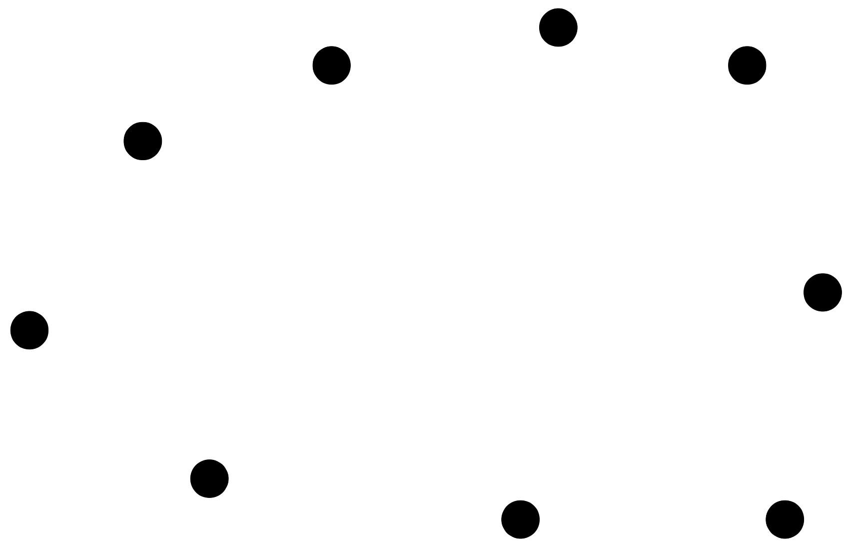


Is there a planar straight-line drawing of a 3-connected graph on P?

3-regular / cubic

Question

point set P in the plane in general position; n := |P| > 3



No!

- otherwise, augment to triangulation
- every triangulation of a convex polygon has a degree-2 vertex

Is there a planar straight-line drawing of a 3-connected graph on P?

3-regular / cubic

Straight-Line Graphs on P

- n = |P|
- h = # boundary vertices of convex hull
- i = # inner vertices of convex hull

k	Necessary and sufficient conditions for a k-connected plane graph	Necessary and sufficient conditions for a k-edge-connected plane graph	Necessary and sufficient conditions for a k-regular plane graph
0			
1			
2			
3			

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1	none	none	n even
2	none	none	none
3	P not in convex position	P not in convex position	

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0	none	none	none	0
1	none	none	n even	$n-1$
2	none	none	none	n
3	P not in convex position	P not in convex position	? (known for $h \leq \frac{3}{4}n$)	$max(\frac{3}{2}n, n+h-1)$

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5	?	?	?	?

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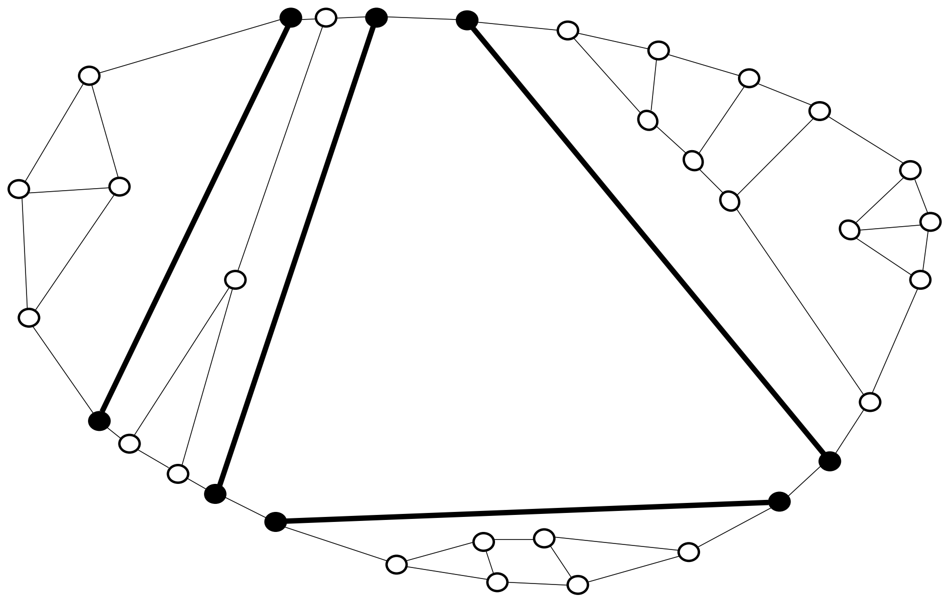
Is there a polynomial time algorithm that finds a cubic graph on a given point set P (if exists)?

Diagonals

From now on: n even, h > 3n/4

- A diagonal of P joins two non-consecutive points of H.



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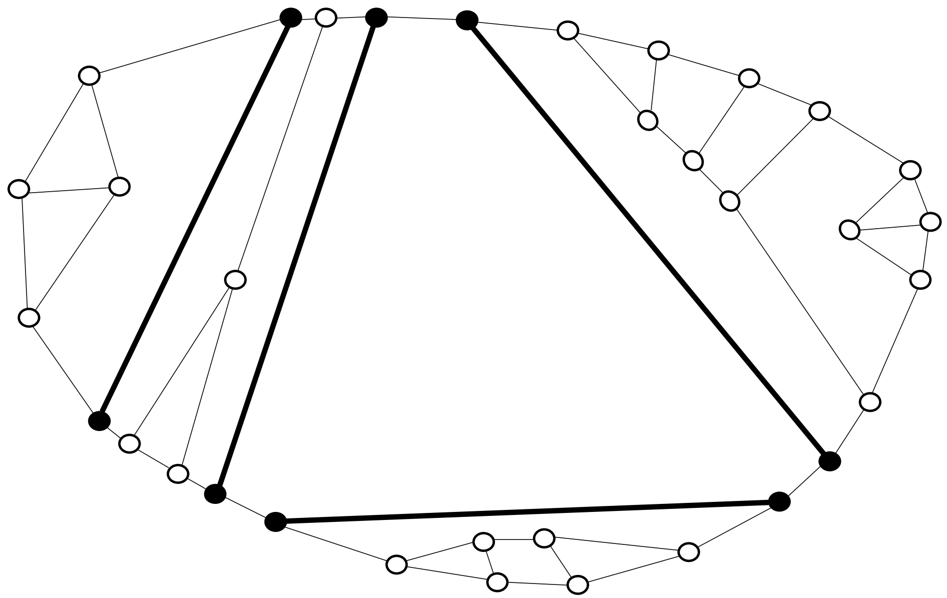
Lemma: Any cubic plane graph on P has at least (h-3i)/2 diagonals.



Diagonals

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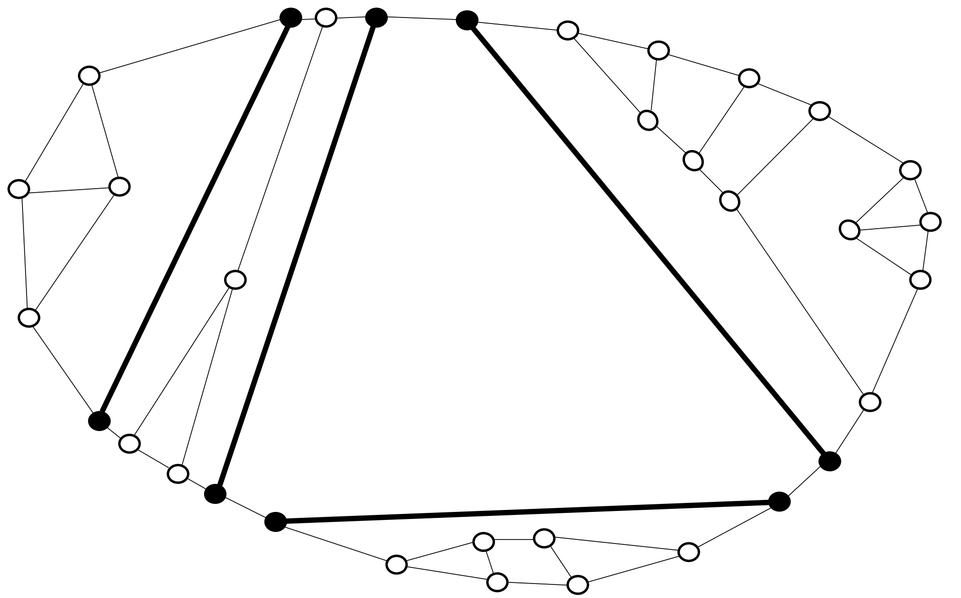
Lemma: Any cubic plane graph on P has at least $(h-3i)/2$ diagonals.

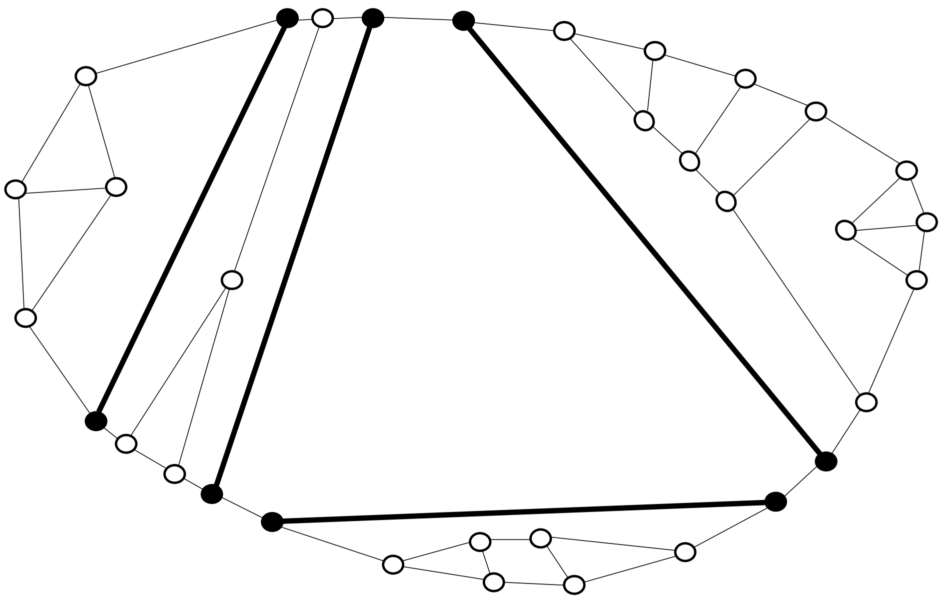
Proof:

- Let s be the number of edges with exactly one vertex in H.
- Let d be the number of diagonals.
- $s \leq 3i$
- $s \geq h-2d$



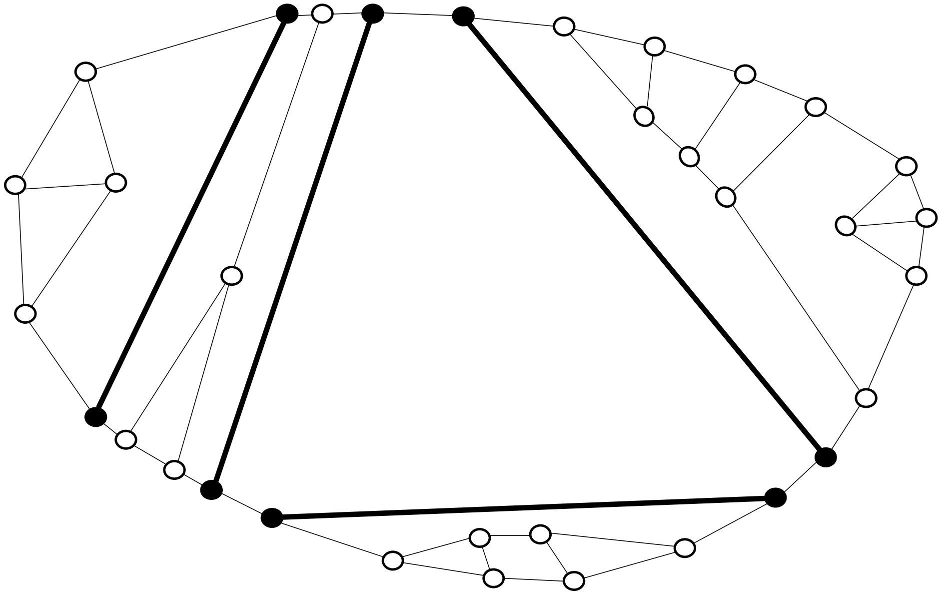
Diagonal Configurations

- Non-crossing diagonals induce a set of induced faces.



Diagonal Configurations

- Non-crossing diagonals induce a set of induced faces.
- Every induced face defines sets I_f, H_f and V_f.

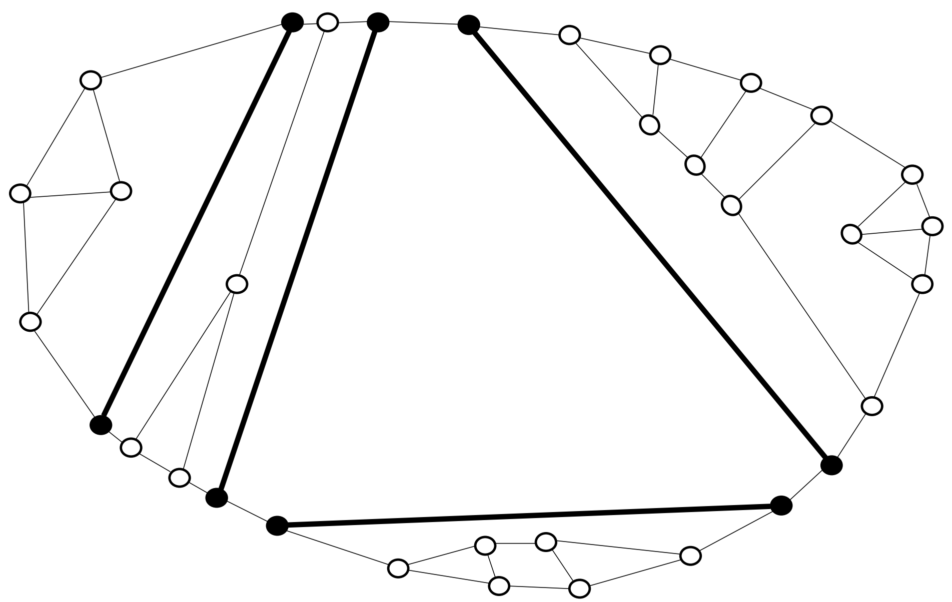


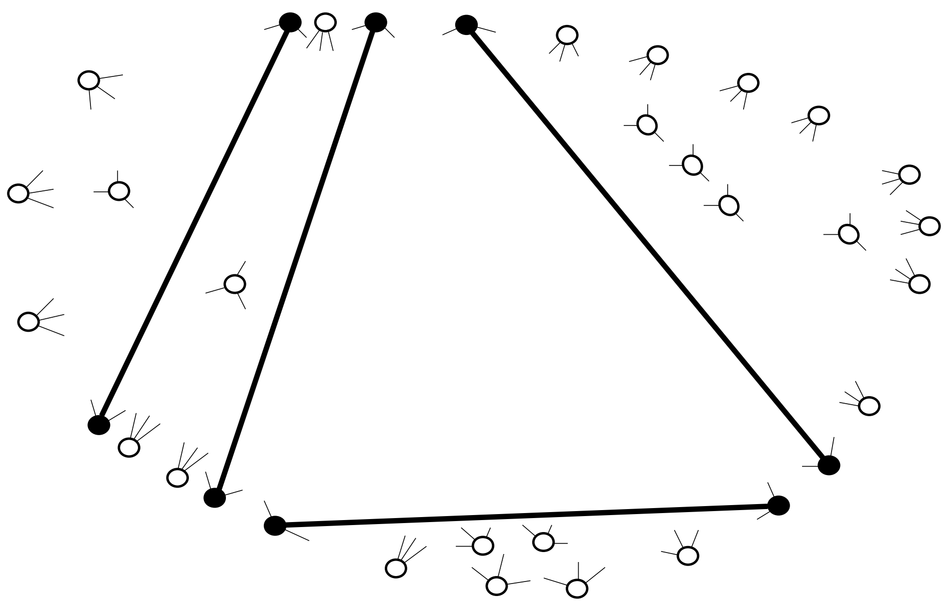
Diagonal Configurations

- Non-crossing diagonals induce a set of induced faces.
- Every induced face defines sets I_f, H_f and V_f.

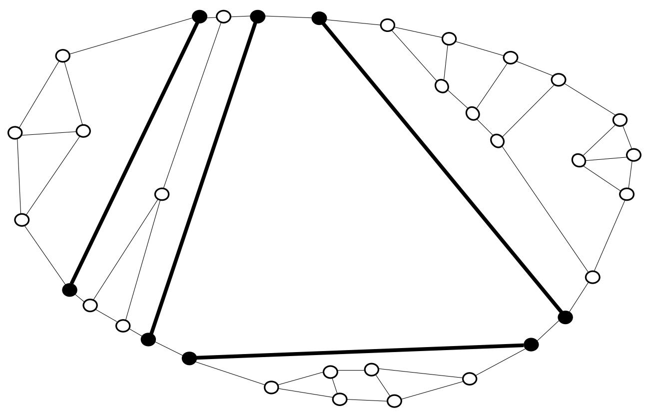
Diagonal configuration = A set of non-crossing diagonals and a multiset of half-edges s.t.

- every point in P has degree 3,
- each half-edge is assigned to an adjacent induced face.

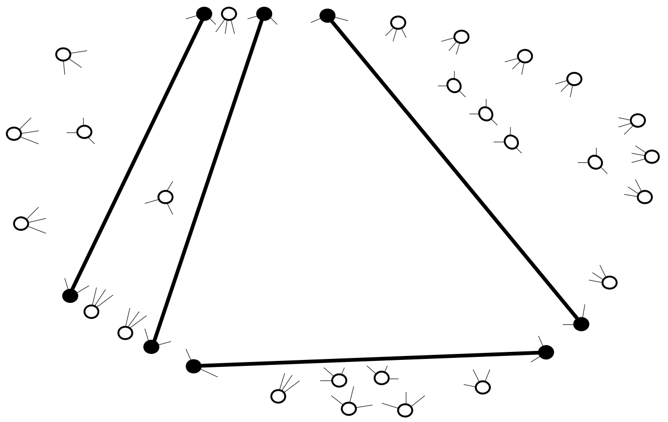




Diagonal Configurations



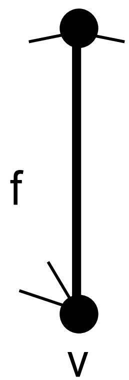
=>



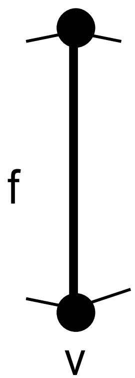
Clear: cubic plane graph => diagonal configuration

Vertices in Diagonal Configurations

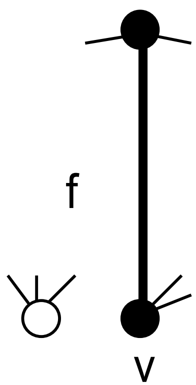
In an induced face f...



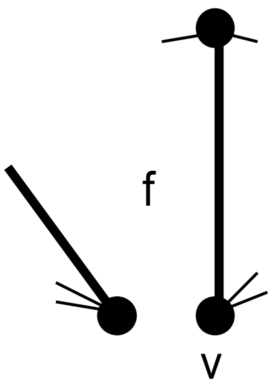
v is hungry ($v \in V^+$)



v is balanced ($v \in V^0$)



v is sated and unmatched ($v \in V^{-u}$)



v is sated and matched ($v \in V^{-m}$)

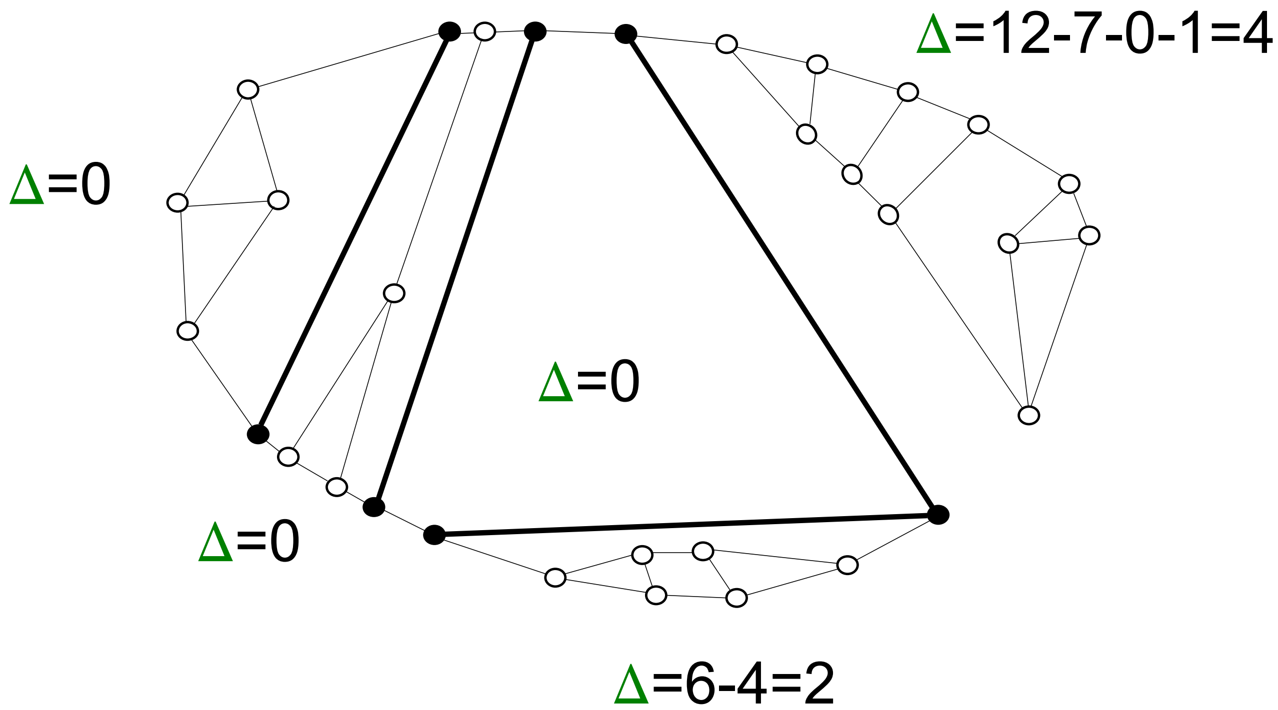
Diagonal Configurations

We define $\Delta(f) = 3i_f - h_f - v_f^{+} - v_f^{-u}$ for every induced face f.

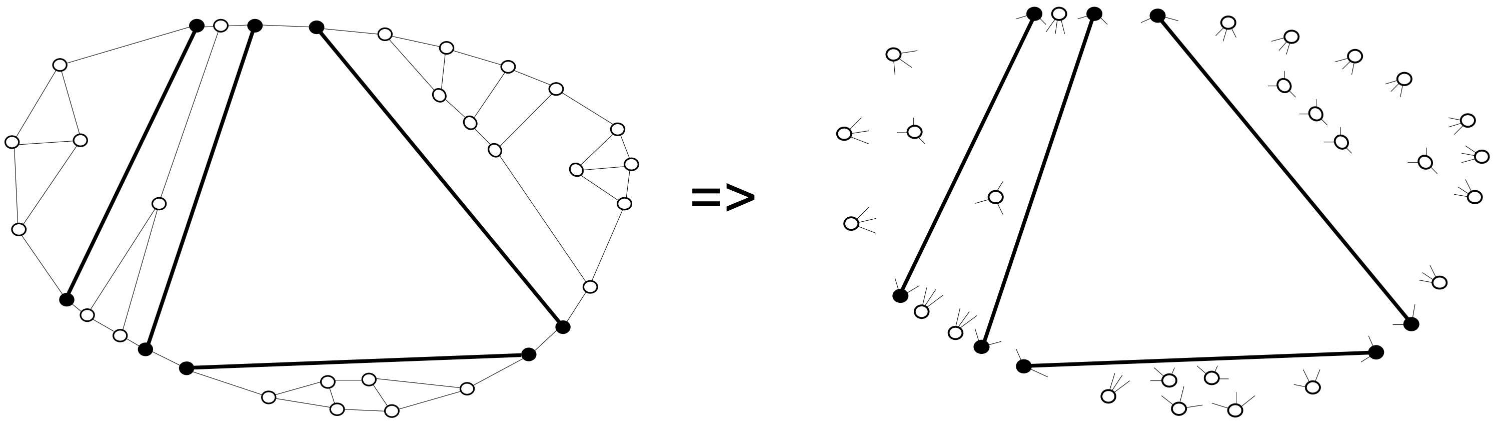
For a cubic graph on P, we have for every f

Lemma: $\Delta(f) \geq 0$

Lemma: $\Delta(f)$ is even



Special Diagonal Configurations



Shown: cubic plane graph => diagonal configuration

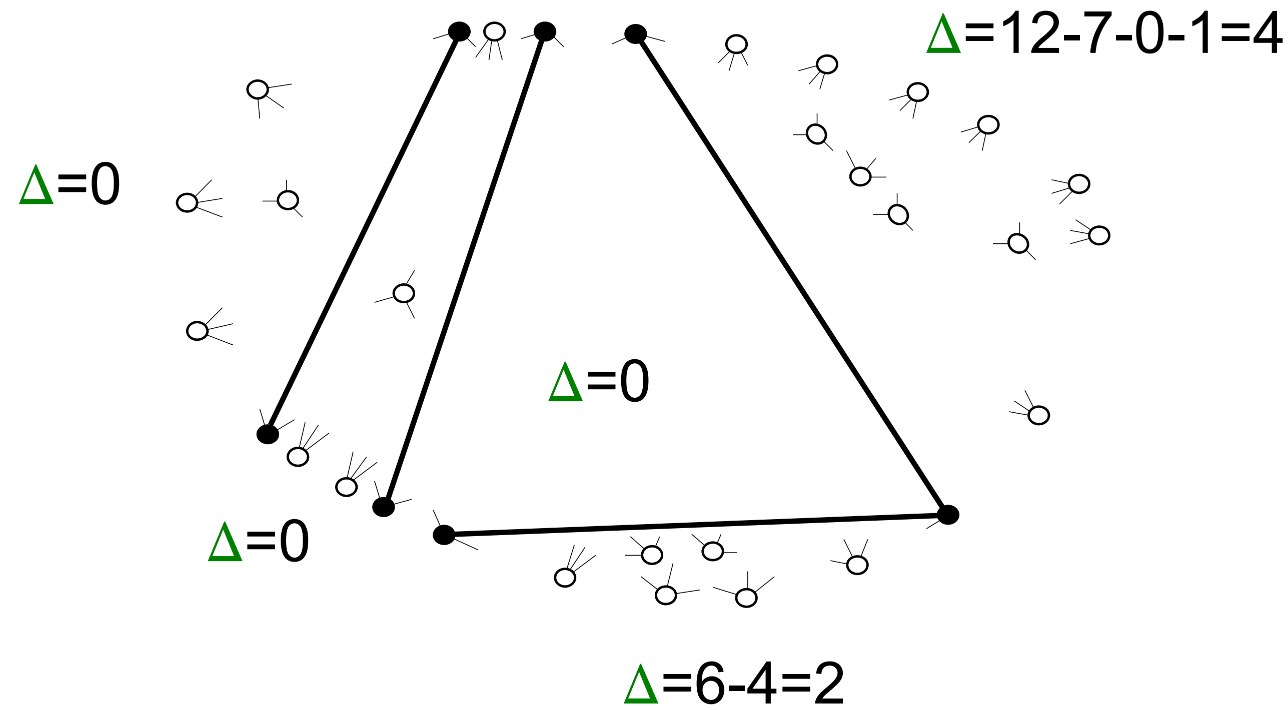
Outline:
diagonal configuration => special diagonal configuration => cubic plane graph (constructively)

This way we can expect “special” cubic graphs on P (if there is any).

Special Diagonal Configurations

We create a special diagonal configuration by applying two operations.

Operation 1: If $\Delta(f) > 0$, cut any boundary edge of f into two half-edges and assign them to the new face.



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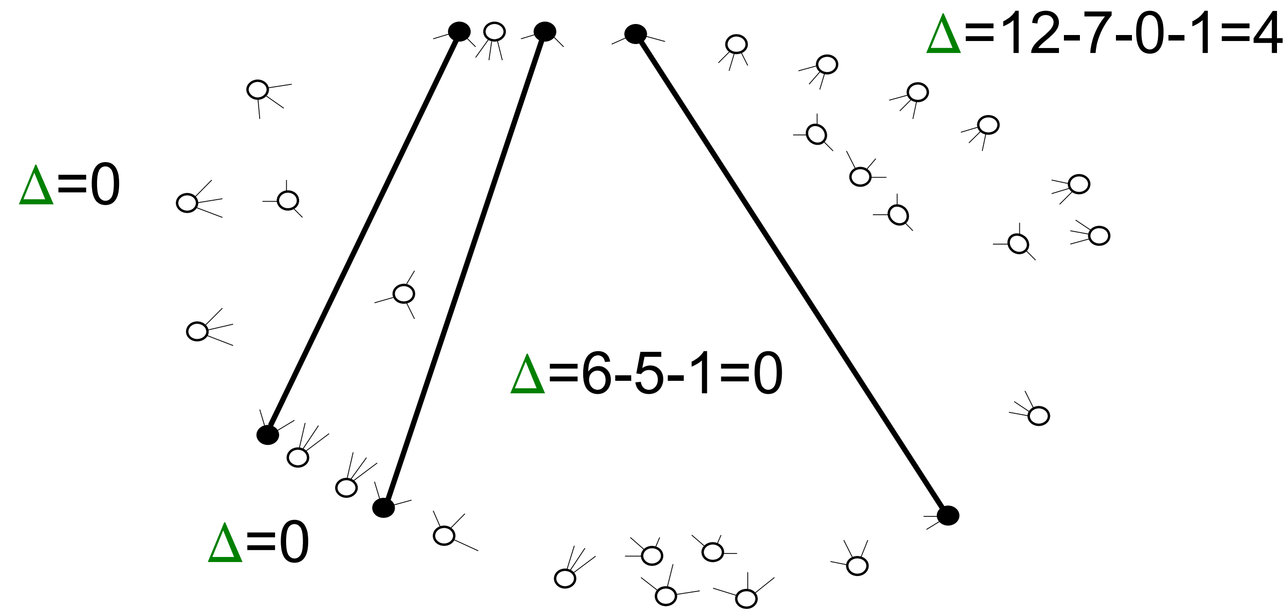
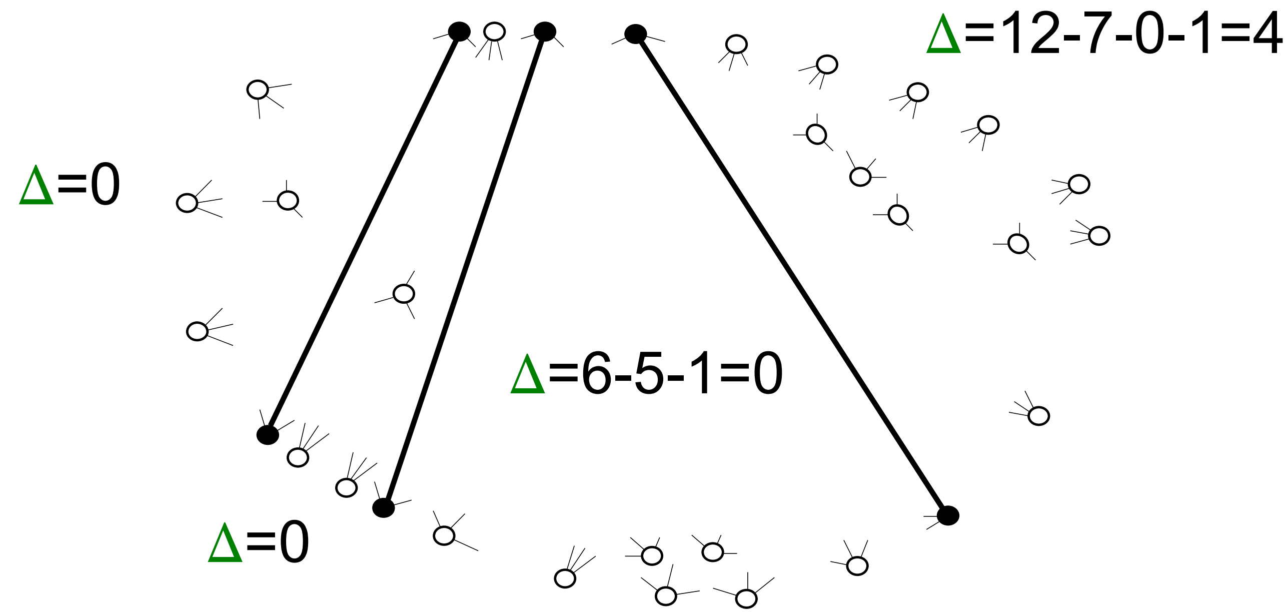


...preserves $\Delta(f) \geq 0$ and $\Delta(f)$ to be even

Special Diagonal Configurations

Operation 2: If $v \in V^{-u}$ for an induced face f, cut the boundary diagonal vw of f into two half-edges and assign them to the new face.



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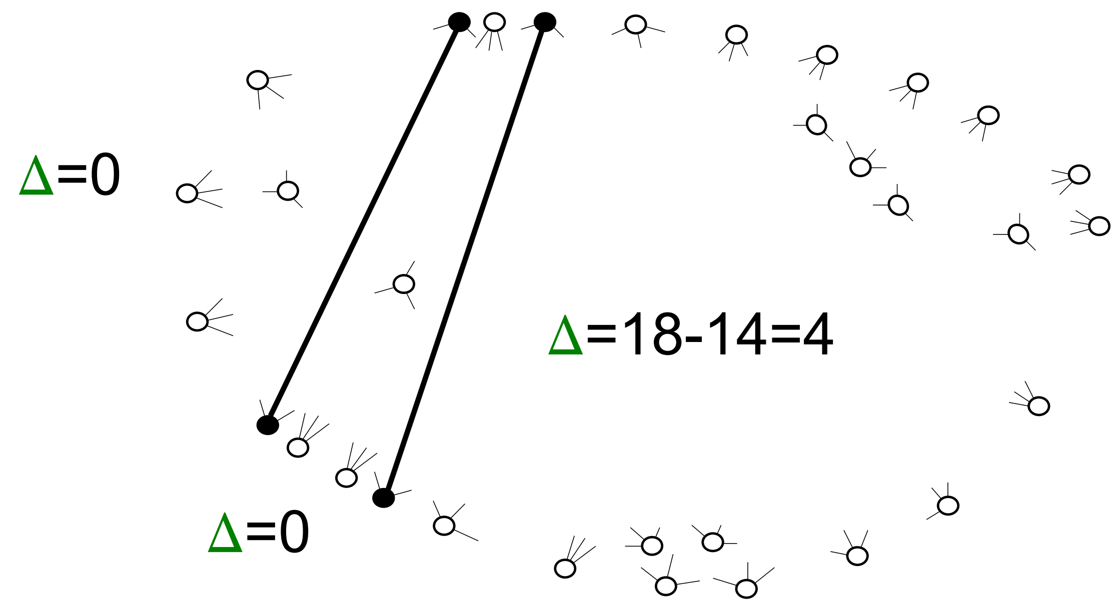


...also preserves Δ(f) ≥ 0 and Δ(f) to be even

Special Diagonal Configurations

Iteratively applying Operations 1 and 2 gives a diagonal configuration s.t. for every induced face f

- $\Delta(f) = 0$ and
- no vertex is unmatched.

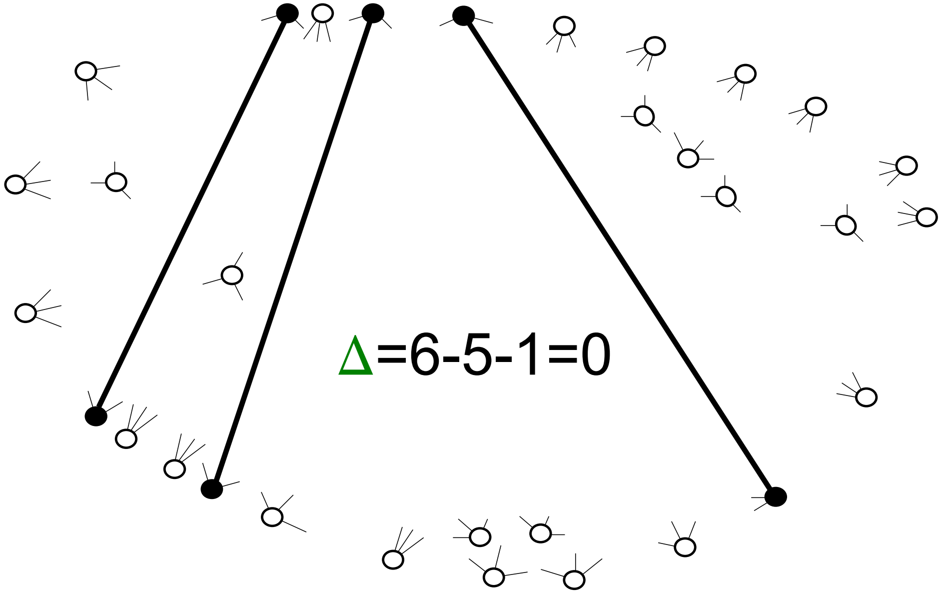
Construction

Let C be a special diagonal configuration.

Lemma: There is a O(n log n) time algorithm constructing a cubic plane graph from C (with no edge joining two inner vertices).

Idea: Create a collection L of 3-stars for each induced face f s.t.

- their union is plane,
- every inner point has degree 3,
- the union of all leaves are exactly the boundary vertices needing a half-edge to I_f.



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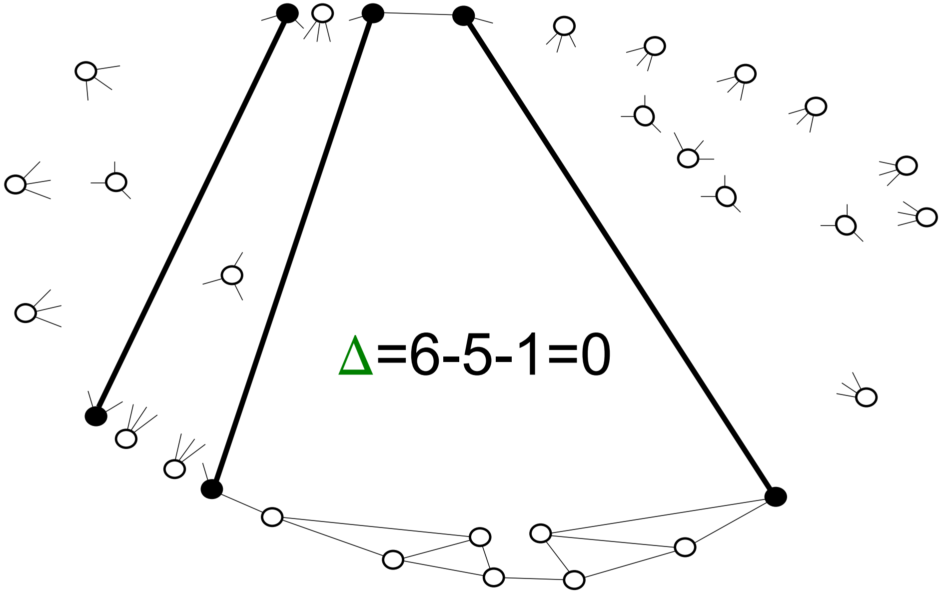

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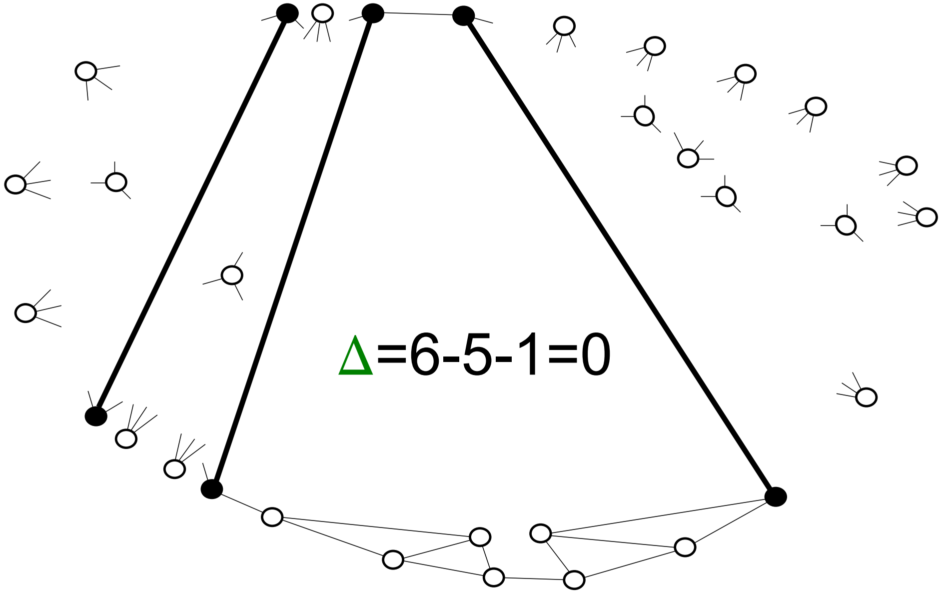

Use Ham-Sandwich cuts (O(n)).

Construction

Thm.: P admits a cubic plane graph if and only if
$h \leq 3n/4$ or there is a special diagonal configuration.

Thm.: P admits a 2-connected cubic plane graph if and only if
$h \leq 3n/4$ or there is a special diagonal configuration and all vertices are balanced.

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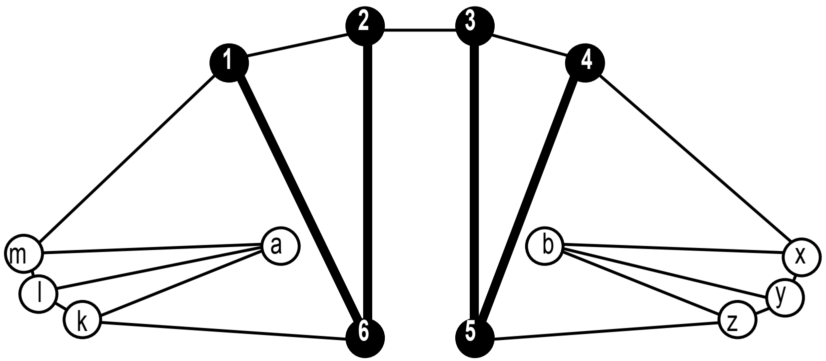
Thm.: P admits a 2-connected cubic plane graph if and only if
$h \leq 3n/4$ or there is a special diagonal configuration and all vertices are balanced.

Thm.: There is a $O(n^3)$ algorithm that constructs a cubic graph on P if possible.

Open Problems

- [Solved] We know a point set, which admits a connected but no 2-connected cubic plane graph. What about 0- and 1-connectivity?



- Augmentation: Given P and a subgraph G on P, is it possible to augment G to a cubic plane graph?

- 4-regular graphs? What about your favorite graph class?