Cubic Plane Graphs on a Given Point Set

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Abstract

Let P be a set of $n \geq 4$ points in the plane that is in general position and such that n is even. We investigate the problem whether there is a cubic plane straight-line graph on P. No polynomial-time algorithm is known for this problem. Based on a reduction to the existence of certain diagonals of the boundary cycle of the convex hull of P, we give the first polynomial-time algorithm; the algorithm is constructive and runs in time $O(n^3)$. We also show which graph structure can be expected when there is a cubic plane graph on P; e. g., if P admits a 2-connected cubic plane graph, we show that P admits also a 2-connected cubic plane graph that contains the boundary cycle of P. The algorithm extends to checking P on admitting a 2-connected cubic plane graph.

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1 Introduction

Let P be a point set in the plane that is in *general position*, i. e., that does not contain three points on a line. A *straight-line embedding* of a graph $G = (V, E)$ is an injective function π: $V \to \mathbb{R}^2$ such that for any two distinct edges ab and cd the straight line segments $\overline{\pi(a)\pi(b)}$ and $\overline{\pi(c)\pi(d)}$ are internally disjoint (i. e., they may only intersect at their endpoints). Let P *admit* a graph $G = (V, E)$ if $|P| = |V|$ and there is a straight-line embedding that maps V to P; we also say that G *is on* P. Thus, P can only admit plane graphs.

We are interested in classifying the point sets P that admit at least one simple plane graph G with a given additional property, e. g., being k-connected, k-edge-connected or k-regular. The graph G is not part of the input: it suffices to find any graph G on P with the desired properties. Using Euler's formula, none of these properties can exist for $k \geq 6$, so we focus on $k \in \{0, \dots, 5\}$. In addition, there are no k-regular graphs for $k = 1, 3, 5$ when n is odd, since every graph must have an even number of odd vertices. We will assume in these cases that n is even. Since we are dealing only with simple graphs, we can further assume $n > k$ throughout the paper. If not stated otherwise, all graphs are assumed to be simple and plane, but not necessarily connected. All point sets are assumed to be in general position.

For $k \in \{0, 1\}$, it is easy to see that every point set admits a 0-connected 0-regular as well as a connected graph, but a 1-regular graph can only be found when n is even (see Table 1). For $k = 2$, every point set P admits a 2-regular 2-connected (and thus also 2-edge-connected) graph, as there is always a plane cycle on P [4].

k	Necessary and sufficient conditions for a k-connected plane graph	k-edge-connected plane graph	k-regular plane graph	Minimum number of edges in a k-(edge-)connected plane graph on P
0	none	none	none	0
1	none	none	n even	$n-1$
2	none	none	none	n
3	P not in convex position	P not in convex position	? (known for $h \leq \frac{3}{4}n$)	$max(\frac{3}{2}n, n+h-1)$
4	? (known for $h = 3$)	?	?	?
5	?	?	?	?

Table 1: Conditions on P that are both necessary and sufficient for the existence of a k-connected, k-edge-connected and k-regular plane graph, respectively, on a point set P in general position, where $|P| = n > k$.

For $k = 3$, Dey et al. [2] give a construction proving that there is a 3-connected graph on P if and only if $n > 3$ and P is not in convex position (the same characterization holds for 3-edge-connected graphs). García et al. [3] investigate how many edges are sufficient to allow a 3-connected graph on P. Let h be the number of points on the convex hull boundary of P. If P is not in convex position, they give a construction of a 3-connected graph on P that has $max(\frac{3}{2}n, n + h - 1)$ edges; the same construction on minimality constraints holds for 3-edge-connected graphs. They also prove that this number is minimal for any 3-connected and for any 3-edge-connected graph on P.

As a corollary, $h \leq \frac{n}{2} + 1$ implies the existence of a 3-regular graph on P. García et al. show in addition that there is a 3-regular graph on P (not necessarily 3-connected, but still 2-connected) when $\frac{n}{2} + 1 \leq h \leq \frac{3}{4}n$ [3, Theorem 4]. While this gives a characterization of the point sets admitting 3-regular graphs for $h \leq \frac{3}{4}n$, the problem remained open for higher values of h. Examples show that the existence of a 3-regular graph is then not any more dependent on only h and n [3]. We give a characterization for all values of h, which leads to the first polynomial time algorithm that computes a 3-regular graph on P if it exists; the running time is $O(n^3)$. We also show that the existence of a 2-connected cubic graph on P implies that there is also a 2-connected cubic graph on P that contains the boundary cycle of the convex hull of P.

In contrast to $k \leq 3$, very little is known about the case $k \in \{4, 5\}$. There exist point sets that are neither in convex position nor admit 4-connected graphs. For the special case of $h = 3$, Dey et al. [2] could characterize the point sets that admit 4-connected graphs.

Preliminaries. A graph is *cubic* if it is 3-regular. Let P be a set of $n \geq 4$ points in the plane in general position with n even. Let $ch(P)$ denote the boundary cycle of the convex hull of P and let a (combinatorial) edge in $\binom{P}{2}$ be a *diagonal* of P if it joins two non-consecutive points in $ch(P)$. Let H be the set of points in $ch(P)$, $h := |H|$, and let $I = P \setminus H$ be the set of inner points in P, $i := |I|$.

Let D be a set of non-crossing diagonals of P. We call the bounded regions in which $ch(P)$ and D subdivides the plane *faces induced by* D; let $F(D)$ be the set of faces induced by D. For every induced face $f \in F(D)$, let V_f be the set of endpoints of diagonals on the boundary of f, let H_f be the set of points on the boundary of f that are not in V_f and let I_f be the set of points strictly inside f. Moreover, let $i_f := |I_f|$, $h_f := |H_f|$, $v_f := |V_f|$, $N_f := H_f \dot{\cup} I_f$ and $n_f := |N_f|$.

For a vertex $v \in V(G)$ and $X \subseteq V(G)$ of a graph G, let $G[X]$ be the subgraph of G that is *vertex-induced* by X. For a simple cubic plane graph G on P and an induced face f of $D(C_G)$, let $G[f] = G[N_f \cup V_f]$.

2 Necessary Conditions for Cubic Graphs

Every simple cubic plane graph G on P contains a (possibly empty) set D of non-crossing diagonals on P. This way, G naturally defines a set of induced faces. Instead of working with concrete cubic graphs, we will consider only their set of diagonals and, for any non-diagonal edge e, the induced face that contains e. It will turn out that it is only important how many non-diagonal edges in an induced face f are incident to a vertex $p \in P$; the precise neighbors of p in f are not crucial. We model this by representing non-diagonal edges with *half-edges*, i. e., with the two parts generated by cutting an edge.

Definition 1. A *diagonal configuration* $\mathcal{C}$ on P consists of a set $D(C)$ of pairwise non-crossing diagonals of P and a multiset of half-edges on the points of P such that, for every vertex $p \in P$,

- p has degree 3 in C (counting diagonals and half-edges) and
- each half-edge on p is assigned to a face that is induced by $D(C)$ and contains p.

Every simple cubic plane graph G on P determines a unique diagonal configuration C_G by cutting every non-diagonal edge e into two half-edges such that both half-edges are assigned to the induced face that contained e. We list necessary conditions on graphs and diagonal configurations to allow cubic graphs on P. García et al. proved the following theorem.

Theorem 2 ([3], implicitly)**.** *Let P be a set of $n \geq 4$ points in general position such that n is even. If $h \leq \frac{3}{4}n$, there is an algorithm with running time $O(n^3)$ that constructs a simple cubic 2-connected plane graph on P that contains $ch(P)$. If $h = n$, P does not admit a simple cubic plane graph.*

We can therefore focus on the case $h > \frac{3}{4}n$; then, every cubic plane graph on P must contain at least one diagonal.

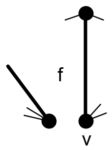
Lemma 3. *Let $h > \frac{3}{4}n$. For every simple cubic (not necessarily connected) plane graph G on P, $D(C_G) \neq \emptyset$. Moreover, $|D(C_G)| \geq 2(h - \frac{3}{4}n) = \frac{h-3i}{2}$.*

Proof. Every vertex v in $H \setminus V(D(C_G))$ has at least one neighbor in I, as v can have at most two neighbors in H and has degree 3. Let s be the number of edges in G that join vertices of H with vertices of I. Then $h - 2|D(C_G)| \leq s$. However, $s \leq 3i$, because every inner vertex can be adjacent to at most 3 vertices in H. It follows that $|D(C_G)| \geq \frac{h-3i}{2} = 2(h - \frac{3}{4}n)$. If $h > \frac{3}{4}n$, then the right-hand side is positive, which concludes the proof. □

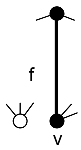
To establish further conditions on diagonal configurations for cubic graphs, we classify the vertices in each induced face f. A vertex in V_f can be one of the following types with respect to f (see Figure 1).

Definition 4. Let C be a diagonal configuration on P and let $f \in F(D(C))$. We call a vertex $v \in V_f$ *sated*, *balanced* or *hungry* (*in* f) if its number of incident diagonals and half-edges in f is 1, 2 or 3, respectively.

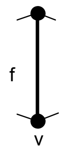
A vertex in V_f that is not balanced is also called *unbalanced*. Let V_f^0, V_f^+ and V_f^- be the set of *balanced*, *hungry* and *sated* vertices in V_f, respectively. For a sated vertex $v \in V_f^-$, let w be the unique neighbor of v in $ch(P)$ that is contained in f. If w is also sated in f, v is called *matched* (with w); otherwise, v is *unmatched*. Let V_f^{-m} and V_f^{-u} be the set of *matched* and *unmatched* vertices in V_f, respectively. Let $v_f^0 := |V_f^0|$, $v_f^+ := |V_f^+|$, $v_f^- := |V_f^-|$, $v_f^{-m} := |V_f^{-m}|$ and $v_f^{-u} := |V_f^{-u}|$.



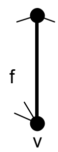
(a) $v \in V_f^{-m}$ is sated (and matched)



(b) $v \in V_f^{-u}$ is sated (and unmatched)



(c) $v \in V_f^0$ is balanced



(d) $v \in V_f^+$ is hungry

Figure 1: Diagonal configurations that contain balanced and unbalanced vertices v (with respect to the induced face f).

We now assign each induced face f the following integer value $\Delta(f)$ in order to prove necessary conditions on these values for cubic graphs.

Definition 5. For a diagonal configuration C on P and each induced face $f \in F(D(C))$, let $\Delta(f) = 3i_f - h_f - v_f^+ - v_f^{-u}$.

It is useful to imagine $h_f + v_f^+ + v_f^{-u}$ as the number of edges between inner and boundary vertices of f that are necessary for any cubic graph on P. Note that an unmatched vertex requires such an edge indirectly, as it forces its non-sated neighbor to require one additional such edge.

Lemma 6. *Let G be a simple cubic plane graph on P. For every face $f \in F(D(C_G))$, $\Delta(f) \geq 0$.*

Proof. Let f be a face in $F(D(C_G))$. We count the number s_f of edges in G that are incident with exactly one vertex in I_f. Each of the v_f^+ hungry vertices in V_f is incident to at least one such edge. There is no diagonal of $H_f \cup V_f$ in G, as otherwise this diagonal would be a diagonal of P due to $H_f \cup V_f \subseteq H$ and contradict f to be an induced face. Therefore, every vertex v in H_f has at most two neighbors in $H_f \cup V_f$ and, hence, at least one neighbor in I_f. Each unmatched vertex $v \in V_f$ has a unique neighbor w in $ch(P) \cap H_f$ such that $vw \notin E(G)$. This forces w to be incident with an additional edge to a vertex in I_f for each such vertex v. It follows that $h_f + v_f^+ + v_f^{-u} \leq s_f$. Since $s_f \leq 3i_f$, $\Delta(f) \geq 0$. □

Let a set of diagonals be *disjoint* if no two of them contain a common end vertex. Note that $D(C_G)$ in Lemma 6 does not have to be disjoint. However, if G contains $ch(P)$, all diagonals are disjoint and consist of balanced vertices only. This gives the following corollary from Lemma 6.

Corollary 7. Let G be a simple cubic plane graph on P that contains $ch(P)$. Then $h_f \leq \frac{3}{4}n_f$ for every induced face $f \in F(D(C_G))$.

We show a necessary parity condition for $\Delta(f)$.

Lemma 8. *Let G be a simple cubic plane graph on P. Then $\Delta(f)$ is even for every induced face $f \in F(D(C_G))$.*

Proof. Consider the graph $G[f]$ and its vertex set $I_f \cup H_f \cup V_f^+ \cup V_f^{-u} \cup V_f^{-m} \cup V_f^0$. By definition, the degree in $G[f]$ of all vertices in V_f^0 is even while the degree of all other vertices in $G[f]$ is odd. As every graph has an even number of odd-degree vertices, $i_f + h_f + v_f^+ + v_f^{-u} + v_f^{-m}$ must be even. However, v_f^{-m} is even, as matched vertices come in pairs. Thus, $i_f + h_f + v_f^+ + v_f^{-u}$ is even and it follows that $3i_f - h_f - v_f^+ - v_f^{-u} = \Delta(f)$ is even. □

3 Constructions

We give sufficient conditions for diagonal configurations to admit cubic graphs. The following result of Tamura and Tamura will be used.

Lemma 9 (Tamura and Tamura [6]). *For points $p_1, p_2, \ldots, p_n$ in general position in the plane and any assignment of degrees from $\{1, 2, \ldots, n-1\}$ to the points such that the sum of degrees is $2n-2$, there is a plane tree with these prescribed degrees on $p_1, p_2, \ldots, p_n$. Moreover, the plane tree can be constructed in time $O(n \log n)$ [1, 3].*

We call an induced face f *empty* if $i_f = h_f = 0$.

Lemma 10. *Let C be a diagonal configuration without unmatched vertices such that for every non-empty induced face $f \in F(D(C))$, $\Delta(f)$ is even and $0 \leq \Delta(f) < 2i_f$. Then there is a simple cubic plane graph G on P with $D(C_G) = D(C)$. If $D(C)$ is additionally disjoint, there is a simple cubic 2-connected plane graph G on P with $D(C_G) = D(C)$ that contains the boundary cycle of P.*

Proof. The proof for the first claim builds on a construction given in [3], but avoids the creation of additional diagonals. We omit the proof in this extended abstract, but state an important structure that is used. Let G' be the graph that consists of $D(C)$ and of all edges in $ch(P)$ that do not join a matched vertex pair. In each induced face $f \in F(D(C))$, we augment G' by a collection L of plane trees, each on at least three vertices, such that (1) the union of all trees is plane, (2) every vertex v in $I_f \cup H_f \cup V_f^+$ is contained in exactly one tree $T \in L$ and (3) v has degree 3 in T if $v \in I_f$ and degree 1 in T if $v \in H_f \cup V_f^+$. This constructs a cubic graph. Note that the precondition of having no unmatched vertex is crucial, as otherwise boundary vertices of f may need more than one incident edge to the interior of f. □

Remark. Intuitively, the precondition $\Delta(f) < 2i_f$ in Lemma 10 avoids that we have to build cycles in the induced face f. If $\Delta(f) \geq 2i_f$, the desired sum of degrees for L would exceed the sum of degrees of every forest in f. The precondition is tight for the statement of Lemma 10, as there are counterexamples even for $\Delta(f) = 2i_f$ ($\Leftrightarrow i_f = h_f + v_f^+$): E. g., in Figure 2, each of the vertices a, b and c has to be adjacent with exactly one of the 3 black vertices for a cubic graph in face f. Thus, a, b and c must form a triangle, which forces every possible edge from b to the boundary to induce a crossing.

For the special case that $\Delta(f) = 0$ for every face f in Lemma 10, each tree will be a $K_{1,3}$. We give an efficient algorithm to construct the trees for this case.

Lemma 11. *Let C be a diagonal configuration without unmatched vertices and $\Delta(f) = 0$ for every induced face $f \in F(D(C))$. There is a $O(n \log n)$ algorithm that constructs a simple cubic plane graph G on P with $D(C_G) = D(C)$ and no edge that joins two points of I.*

Proof. We create a graph G' and, for every induced face $f \in F(D(C))$, a collection L of trees that satisfy the properties (1)–(3), as described in the proof of Lemma 10. As $\Delta(f) = 0$, the number of vertices in $H_f \cup V_f^+$ for every face f is exactly $3i_f$ and every such vertex needs exactly one additional incident edge to the interior of f. Thus, L must consist of i_f trees, each of which is a $K_{1,3}$. We take two point sets of equal size: one is $Z := H_f \cup V_f^+$ and the other set Y is generated from I_f by replacing each point $p \in I_f$ with three points that are in a sufficiently small ϵ-neighborhood of p.

We need to compute a plane matching between Y and Z. Clearly, it can be assumed that $Y \cup Z$ is in general position. We compute a ham-sandwich cut for Y and Z in time $O(n)$ [5]. Iterating the computation for the subsets of $Y \cup Z$ that are contained in each side of the cut, respectively, will terminate with cells containing exactly one point of Y and one point of Z. Joining the two points for every cell by an edge constructs L, giving a total running time of $O(n \log n)$. $\square$

4 Reduction to Diagonal Configurations

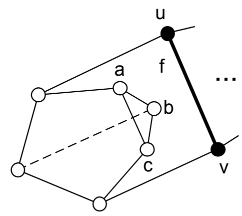


Figure 2: An induced face f with $\Delta(f) = 2i_f = 6$ that does not admit a simple cubic plane graph.

We have shown that a suitable diagonal configuration allows to construct a cubic graph on P. If we can show that every cubic graph on P implies the existence of such a suitable diagonal configuration, we have reduced the problem of computing cubic graphs to diagonal configurations. We will give an efficient algorithm for finding such a diagonal configuration in the next section. Let $h > \frac{3}{4}n$ due to Theorem 2.

Let C_G be the diagonal configuration of a simple cubic plane graph G on P. We will transform C_G to the desired diagonal configuration by iteratively applying two operations. In each step, we maintain a diagonal configuration C, which is initialized with C_G. Note that this does not transform the graph G itself.

Operation 1. Let $\Delta(f) > 0$ for an induced face f of $D(C)$ (see, e. g., any graph in Figure 1). Cut a diagonal vw on the boundary of f into two (non-diagonal) half-edges. Both half-edges are assigned to the newly generated induced face z.

Operation 2. Let $v \in V_f^{-u}$ for an induced face f of $D(C)$ (see, e. g., Figures 3(b) and 3(c)). Cut the unique diagonal vw on the boundary of f into two (non-diagonal) half-edges. Both half-edges are assigned to the newly generated induced face z.

For being able to construct a cubic graph from a diagonal configuration, we need that $\Delta(f) \geq 0$ and $\Delta(f)$ is even for every induced face f. We prove that the above operations preserve these properties of C in every step.

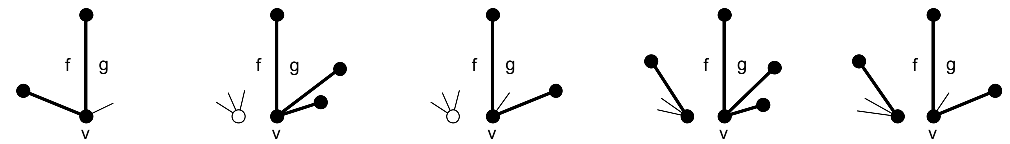
Lemma 12. *Operations 1 and 2 preserve for every induced face f' that $\Delta(f') \geq 0$ and that $\Delta(f')$ is even.*

Proof. Consider the diagonal configuration C before an Operation 1. As $D(C) \neq \emptyset$ due to $h > \frac{3}{4}n$, at least one diagonal vw exists. Let g be the induced face of $D(C)$ that is separated from f by vw. According to Lemmas 6 and 8, $\Delta(f) \geq 2$ and $\Delta(g) \geq 0$. We check the effects of Operation 1 on $\Delta(z)$ in dependence of the type of v. By symmetry, the same effects hold for w. Figure 3 lists the five possible configurations for vertex types of v in f and g; note that this covers all cases, as v cannot be hungry or sated in both faces, respectively. We illustrate the effect on $\Delta(z)$ for the case $v \in V_f^{-m}$ and $v \in V_g^{+}$ (see Figure 3(e)).

Since $v \in V_f^{-m} \cap V_g^{+}$, the contribution of v to $\Delta(g)$ in C is -1, while the contribution of v to $\Delta(f)$ is 0, as matched vertices do not influence Δ. Applying Operation 1 replaces these two contributions by one value dependent on z. If v has been incident to only one diagonal of D, v will be a new vertex in H_z (and, thus, in N_z) after performing Operation 1, causing $\Delta(z)$ to decrease by 1. Otherwise, v will be in V_z^{+}, which also decreases $\Delta(z)$ by 1. Additionally, $\Delta(z)$ decreases by another 1 in both cases, since the vertex in f that was formerly matched to v is now unmatched. In total, the effect of v on $\Delta(z)$ is $1 - 2 = -1$. Table 2 lists the effects for the other four cases.

$v \in$	$\Delta(f)$	$v \in$	$\Delta(g)$	$v \in$	$\Delta(z)$	effect on $\Delta(z)$	Fig.
V_f^0	$+0$	V_g^0	$+0$	$H_z \cup V_z^+$	-1	-1	3(a)
V_f^{-u}	-1	V_g^0	$+0$	V_z^0	$+0$	$+1$	3(b)
V_f^{-u}	-1	V_g^+	-1	$H_z \cup V_z^+$	-1	$+1$	3(c)
V_f^{-m}	$+0$	V_g^0	$+0$	V_z^0	$-1^{(\star)}$	-1	3(d)
V_f^{-m}	$+0$	V_g^+	-1	$H_z \cup V_z^+$	$-2^{(\star)}$	-1	3(e)

Table 2: The five possible configurations of a diagonal end vertex v before Operation 1. Columns 2 and 4 depict the contribution of v to the given value. Column 6 depicts the contribution of v to $\Delta(z)$ after Operation 1. For entries marked with $^{(\star)}$, the sated vertex in f that was matched to v is not matched anymore in z.



(a) v is in V_f^0 and V_g^0 (b) v is in V_f^{-u} and V_g^0 (c) v is in V_f^{-u} and V_g^+ (d) v is in V_f^{-m} and V_g^0 (e) v is in V_f^{-m} and V_g^+

Figure 3: Possible initial configurations for v.

According to Table 2, $\Delta(z) = \Delta(f) + \Delta(g) + x$ with $x \in \{-2, 0, 2\}$ after applying Operation 1, since exactly two vertices change. This implies $\Delta(z) \geq 0$ and that $\Delta(z)$ is still even.

Consider the diagonal configuration C before an Operation 2. As $v \in V_f^{-u}$, v must be either contained in V_g^0 or in V_g^+ (see Figures 3(b) and 3(c)). In both cases, the effect of v on $\Delta(z)$ is $+1$, according to Table 2. Thus, after applying Operation 2, $\Delta(z) = \Delta(f) + \Delta(g) + x$ with $x \in \{0, 2\}$ and the claim follows. □

We are now able to state our main structural results.

Theorem 13. *The following statements are equivalent:*

(1) *P admits a simple cubic plane graph G.*

(2) *$h \leq \frac{3}{4}n$ or there is a diagonal configuration C on P such that $\Delta(f) = 0$ for every induced face $f \in F(D(C))$ and no vertex is unmatched.*

Proof. The proof for *(2)* ⇒ *(1)* follows directly from Theorem 2 for $h \leq \frac{3}{4}n$ and from Lemma 11 for $h > \frac{3}{4}n$. We prove *(1)* ⇒ *(2)*. Assume that $h > \frac{3}{4}n$. Then $D(C_G) \neq \emptyset$ with Lemma 3 and we can iteratively apply (any of the) Operations 1 and 2 on C_G as long as possible until the process terminates with a diagonal configuration C'. Note that the application of each operation decreases the number of diagonals by one; thus, $D(C') \subseteq D(C_G)$. Due to Lemma 12, every induced face $f \in F(D(C'))$ satisfies $\Delta(f) = 0$ and no vertex can be unmatched. $\square$

Corollary 14. If $h > \frac{3}{4}n$, every simple cubic plane graph on P implies the existence of a simple cubic plane graph on P that does neither contain an unmatched vertex nor an edge joining two vertices in I.

Theorem 15. *The following statements are equivalent:*

(1) *P admits a simple cubic 2-connected plane graph G.*

(2) *P admits a simple cubic 2-connected plane graph G' such that G' contains $ch(P)$ and the diagonals $D(C_{G'})$ are disjoint.*

(3) *$h \leq \frac{3}{4}n$ or there is a diagonal configuration C on P such that there are no unbalanced vertices and $h_f = \frac{3}{4}n_f$ for each induced face $f \in F(D(C))$.*

Proof. The proof for *(2)* $\Rightarrow$ *(1)* is immediate. We prove *(3)* $\Rightarrow$ *(2)*. In case that $h \leq \frac{3}{4}n$, Theorem 2 settles the claim. Let $h > \frac{3}{4}n$. Then $D(C)$ must be disjoint, as every vertex v that is end point of two diagonals would contradict that v is balanced in every induced face. With $h_f = \frac{3}{4}n_f$ and $d_f^+ = d_f^- = 0$ for every induced face f, $\Delta(f) = 0$ follows. Applying the construction of Lemma 11 yields the desired graph G' and ensures $ch(P) \subseteq G'$, as $D(C)$ is disjoint.

It remains to prove *(1)* $\Rightarrow$ *(3)*. We can assume $h > \frac{3}{4}n$. Then G contains at least one diagonal with Lemma 3. As any unbalanced in C_G would imply G to contain a cut vertex, there are no unbalanced vertices in C_G. Iteratively applying Operation 1 on C_G results in a diagonal configuration that satisfies $\Delta(f) = 0$ for every induced face f. As Operation 1 does not introduce new unbalanced vertices, $h_f = \frac{3}{4}n_f$ for every f, which gives the claim. $\square$

The properties we get from not being able to apply Operation 1 are valid also for cubic plane graphs that contain the least possible number of diagonals for P. This implies for every face f that every vertex in I_f has to be joined to exactly three vertices in H_f. We get the following corollary.

Corollary 16. If $h > \frac{3}{4}n$, every simple 2-connected cubic plane graph on P that contains the least possible number of diagonals contains $ch(P)$.

If we insist on 2-connected cubic graphs that contain $ch(P)$, the number of diagonals in these graphs is completely determined by n and h. Given n and h, the number of diagonals is $2h - \frac{3}{2}n$.

Corollary 17. If $h > \frac{3}{4}n$, every simple cubic 2-connected plane graph on P implies the existence of a simple cubic 2-connected plane graph on P such that $ch(P) \subseteq G$, $D(C_G)$ is disjoint, $|D(C_G)| = 2h - \frac{3}{2}n$ and there is no edge in G that joins two vertices in I.

One could be tempted to prove that every point set P admitting a cubic plane graph also admits a cubic 2-connected plane graph. However, this is not true because of the following counterexample.

Lemma 18. *There is no simple 2-connected cubic plane graph on the point set P in Figure 4.*

Proof. Assume to the contrary there is such a graph G. Since $h > \frac{3}{4}n$, we may assume with Corollary 17 that G contains $ch(P)$ and exactly 3 disjoint diagonals. For every diagonal Z with an end vertex in $\{x, y, z, k, l, m\}$, let $hp(Z)$ be the open halfplane defined by Z that contains not a. Note that $hp(Z)$ does not contain b either. Assuming there is a diagonal ending at a vertex in $\{x, y, z, k, l, m\}$, we choose one such diagonal Z with a minimal number of points in $hp(Z)$. As $hp(Z)$ is non-empty, but contains no inner vertex, every vertex in $hp(Z)$ has degree 2, contradicting the cubicness of G. This leaves the six remaining candidates $\{1, \ldots, 6\}$ for an end vertex of a diagonal. No diagonal can join two vertices of $\{1, \ldots, 4\}$, as these vertices form a path in $ch(P)$. Thus, there can be only two disjoint diagonals that have end vertices 5 and 6, respectively, contradicting that there have to be 3 disjoint diagonals. □

With a bit more effort, one can show that the simple cubic plane graph shown in Figure 4 is the only one possible. Lemma 18 gives rise to the question whether we can expect connectivity on a point set P admitting a simple cubic plane graph. As we could not find an analogue counterexample that distinguishes 0- and 1-connectivity, this is still an open problem.

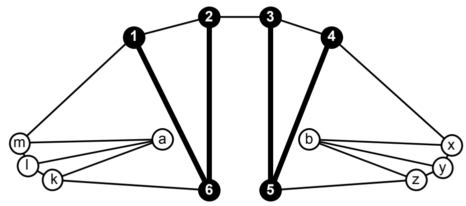


Figure 4: A point set that admits no simple 2-connected cubic plane graph.

5 The Algorithms

In this section we describe two algorithms. We first describe an algorithm for finding a 2-connected cubic plane graph on a given point set P (if it exists). By Theorem 15, it suffices to look for a plane graph G such that it contains all edges of $ch(P)$ and all faces induced by the set of the diagonals of G satisfy $\Delta(f) = 0$. Then we outline a similar, very technical algorithm for finding a cubic plane graph (not necessarily connected) on a given point set P (if it exists). In both cases we use a dynamic-programming approach. For a pair of points a, b, let $R(a, b)$ be the closed halfplane to the right of the line ab (oriented from a to b). For a point $x \in H$, let x^+ and x^- be the points of H counterclockwise and clockwise of x, respectively.

Let T be the set of ordered pairs of distinct non-neighboring vertices of $ch(P)$. During the first algorithm, for each pair $(a, b) \in T$, we compute $d(a, b)$, which is the maximum number of pairwise disjoint diagonals of P such that all these diagonals lie in $R(a, b)$ and all faces induced by them satisfy $\Delta(f) = 0$, with the possible exception for the unique face intersecting the complement of $R(a, b)$.

For $(a, b) \in T$, let $H(a, b)$ be the set of points of H lying in the interior of $R(a, b)$, let $I(a, b)$ be the set of points of I lying in $R(a, b)$, and let $\Delta(a, b) := 3|I(a, b)| - |H(a, b)|$. When computing the numbers $d(a, b)$, we make use of the following observation:

Observation 19. *(i) Let $(a, b) \in T$. Let C be a set of disjoint diagonals in $R(a, b)$, which does not include the diagonal ab. Let f be the unique face induced by C, which contains the segment ab and some points to the right of the segment ab. Then there is a $c \in H(a, b) \cup \{a\}$ such that the face f also contains the pair c, c^+ of adjacent vertices of $ch(P)$. Consequently, every diagonal of C lies entirely in one of the closed halfspaces $R(a, c)$ and $R(c^+, b)$.*

(ii) Let $(a, b) \in T$. Let C be a set of disjoint diagonals in $R(a, b)$, containing the diagonal ab. Then all the other diagonals of C lie in $R(a^+, b^-)$.

Algorithm for finding a 2-connected cubic graph

If n is odd or $n \leq 3$, there is no cubic graph on P; we therefore assume that n is even and $n \geq 4$. If $h \leq \frac{3}{4}n$, we can find a 2-connected cubic plane graph on P in time $O(n^3)$ due to Theorem 2. Let $h > \frac{3}{4}n$. In particular, $h > 3$.

For all pairs $(a, b) \in T$ with $|H(a, b)| \leq 2$, no diagonal in $H(a, b)$ can exist and we set $d(a, b) := 0$. For technical reasons, we also set $d(a, a^+) := d(a, a) := 0$ for each $a \in H$. The values $d(a, b)$ for pairs $(a, b) \in T$ with $|H(a, b)| \geq 3$ are computed in the order of increasing value of $|H(a,b)|$. Thus, we first compute $d(a,b)$ for all pairs $(a,b) \in T$ with $|H(a,b)| = 3$, then for all pairs $(a,b) \in T$ with $|H(a,b)| = 4$, etc.. For each pair $d(a,b)$, we proceed using the following recursion rule.

For a given $(a,b) \in T$ with $|H(a,b)| \geq 3$, let

$$d := \max\{d(a,c) + d(c^+,b) : c \in H(a,b) \cup \{a\}\}.$$

The number d equals $d(a,b)$, unless $d(a,b)$ is witnessed only by diagonal sets containing the diagonal ab. If $d = d(a^+,b^-)$ and $\Delta(a,b) + 2d = 0$, we can add the diagonal ab to the $d(a^+,b^-)$ diagonals, as $\Delta(f) = 0$ for the induced face f in $R(a,b)$ that contains ab. In this case, we set $d(a,b) := d + 1$. Otherwise, we set $d(a,b) := d$. Note that no diagonal in $R(a,b)$ is counted twice by taking $d(a,c) + d(c^+,b)$ in the recursion, as $R(a,c)$ and $R(c^+,b)$ are disjoint. Due to Observation 19, $d(a,b)$ is indeed the maximum number of pairwise disjoint diagonals of P such that all these diagonals lie in $R(a,b)$ and all faces induced by them satisfy $\Delta(f) = 0$, with the possible exception for the unique face intersecting the complement of $R(a,b)$.

After computing $d(a,b)$ for all pairs $(a,b) \in T$ in this way, we check if there is a pair $(a,b) \in T$ satisfying

$$2(d(a,b) + d(b,a)) \geq h - 3i.$$

The computation of $d(a,b)$ for each pair $(a,b) \in T$ takes time $O(n)$. Clearly, all the computation so far can be done in time $O(n^3)$.

Lemma 20. *If $h > 3$ then P admits a 2-connected cubic plane graph if and only if $2(d(a,b) + d(b,a)) \geq h - 3i$ for some pair $(a,b) \in T$.*

Proof. If P admits a 2-connected cubic plane graph, then by Theorem 13, there is a diagonal configuration C on P such that $\Delta(f) = 0$ for every induced face $f \in F(D(C))$ and no vertex is unmatched. Choose any diagonal ab lying in C. By the definition of $d(a,b)$, the number of diagonals of C contained in $R(a,b)$ is at most $d(a,b)$. Similarly, the number of diagonals of C contained in $R(b,a)$ is at most $d(b,a)$. Since each diagonal of C lies in $R(a,b)$ or in $R(b,a)$, we obtain that $d(a,b) + d(b,a) \geq |C|$. Consequently, by Lemma 3, $2(d(a,b) + d(b,a)) \geq |2|C| \geq h - 3i$.

On the other hand, suppose now that $2(d(a,b) + d(b,a)) \geq h - 3i$ for some pair $(a,b) \in T$. Consider the two sets, D_1 and D_2, of diagonals witnessing the values of $d(a,b)$ and $d(b,a)$, respectively. If the diagonal ab lies in both of them, then each face induced by the set $D_1 \cup D_2$ satisfies $\Delta(f) = 0$ and we can find a 2-connected cubic plane graph on P due to Lemma 11. Suppose now that the diagonal ab does not lie in $D_1 \cap D_2$. If it lies neither in D_1 nor in D_2 then $\Delta(f) = 0$ holds for each face induced by the set $D_1 \cup D_2$, with the possible exception for the face containing the segment ab. If the diagonal ab lies exactly in one of the sets D_1 and D_2, then $\Delta(f) = 0$ holds for each face induced by the set $D_1 \cup D_2$, with the possible exception for one face adjacent to the diagonal ab. Due to Lemma 3, removing consequently $\frac{h-3i}{2} - (d(a,b) + d(b,a))$ diagonals at the boundary of the unique face not satisfying $\Delta(f) = 0$ results in a set of diagonals inducing only faces satisfying $\Delta(f) = 0$. Then, due to Lemma 11, there is a 2-connected cubic plane graph on P. □

Thus, if there is no pair $(a,b) \in T$ satisfying $2(d(a,b) + d(b,a)) \geq h - 3i$ then there is no 2-connected cubic plane graph on P. Otherwise, if we find a pair $(a,b) \in T$ satisfying $2(d(a,b) + d(b,a)) \geq h - 3i$, then we use the following recursive procedure which finds diagonals of two sets witnessing the values of $d(a,b)$ and $d(b,a)$, respectively.

Procedure Split(x,y);
If $d(x,y) = d(x^+,y^-) + 1$ and $\Delta(x,y) + 2d(x^+,y^-) = 0$, we add the diagonal xy to D and run Split(x^+,y^-). Otherwise, if $d(x,y) > 0$, we find a $c \in H(x,y) \setminus \{y^-\}$ such that

$$d(x,y) = d(x,c) + d(c^+,y)$$

and then run (for one such c) procedures Split(x, c) and Split(c^+, y).
End of Procedure Split.

The procedure Split follows the way in which the numbers $d(a, b)$ were computed. Therefore, if we first set $D := \emptyset$ and then run procedures Split(a, b) and Split(b, a), we obtain a set D of diagonals such that each face induced by D satisfies $\Delta(f) \geq 0$. This can be done in time $O(n^2)$. We then apply Operation 1 as long as $\Delta(f) > 0$ for some of the induced faces. We obtain a set D' of diagonals such that $\Delta(f) = 0$ holds for each induced face. Then we find pairwise non-intersecting 3-stars in each face covering all inner points in the face according to Lemma 11. These stars in all faces together with the diagonals of D' and with the edges of $ch(P)$ form a 2-connected cubic plane graph on P. The whole algorithm can be done in time $O(n^3)$.

Algorithm for finding any cubic graph

If 2-connectivity is not required then the algorithm is similar but much more involved than the above algorithm for finding 2-connected cubic plane graph. No substantially new ideas are needed, therefore we only give a sketch of the algorithm.

An *H-edge* is an edge connecting a pair of vertices of H.

In its main part, the algorithm computes special quantities $z(a, b, s, t, j)$, where $a, b \in H$ with $a \neq b$, s and t are contained in $\{0, 1, 2, 3\}$ and $j \in \{0, 1\}$. In fact, not all the quantities $z(a, b, s, t, j)$ are computed. Some of them do not make sense, e. g., for parity reasons. The quantity $z(a, b, s, t, j)$, whenever it exists, is equal to the maximum size of a set S of H-edges lying in $R(a, b)$ such that the following three conditions hold:

(i) No four edges of S have a common vertex,

(ii) S contains the diagonal ab if and only if $j = 1$, and

(iii) the diagonals lying in S are exactly the diagonals of some "nearly" diagonal configuration on the point set $P \cap R(a, b)$, meaning that the configuration satisfies all the properties of a diagonal configuration, except that the degrees of a and b are $3 - s$ and $3 - t$, respectively. Moreover, the degrees in each face allow to connect the half-edges into edges such that 3-stars are formed and each 3-star connects at least one vertex of H with a vertex of I.

Note that j is an indicator of the diagonal ab (Condition (ii)) and s and t are "free" degrees (half-edges) to be used for edges going outside of $R(a, b)$.

When computing the numbers $z(a, b, s, t, j)$, we make use of the following analogue of Observation 19:

Observation 21. *Let $(a, b) \in T$. Let C be a set of non-crossing diagonals in $R(a, b)$, which may include the diagonal ab. Let f be the unique face induced by C, which contains the segment ab and some points to the right of the segment ab. Then there is a $c \in H(a, b)$ contained in the face f. Consequently, every diagonal of C other than ab lies entirely in one of the closed halfspaces $R(a, c)$ and $R(c, b)$.*

At the beginning of the algorithm, for each $a \in H$, we set $z(a, a^+, 2, 2, 1) := 1$ and $z(a, a^+, 3, 3, 0) := 0$. All the other values $z(a, a^+, s, t, j)$ are not defined. Then, in the main part of the algorithm, if $z(a, b, s, t, j)$ with $|H(a, b)| \geq 1$ exists then, due to Observation 21, it is determined as

$$\max(j + z(a, c, s', t', j') + z(c, b, s'', t'', j'')),$$

where the maximum is taken over all $c \in H(a, b), s', s'', t', t'', j', j''$, for which the following five conditions hold:

1. $z(a, c, s', t', j')$ and $z(c, b, s'', t'', j'')$ exist,
2. $s + j \leq s'$ (degree condition at a),
3. $t + j \leq t''$ (degree condition at b),
4. $s'' + t' \geq 3$ (degree condition at c),

5. if $j = 1$ then $\Delta'(a, b) + 2z(a, c, s', t', j') + 2z(c, b, s'', t'', j'') = 0$, where $\Delta'(a, b) = 3i - 3h - (3 - s - j) - (3 - t - j)$.

The last condition ensures that we find a diagonal configuration inducing faces with $\Delta(f) \geq 0$. After applying Operations 1 and 2 as long as it is needed, we obtain a diagonal configuration with no unmatched vertex and with $\Delta(f) = 0$ for each induced face. Then we can complete the construction of a plane cubic graph similarly as in the previous algorithm. Again, the whole algorithm runs in time $O(n^3)$. This can be checked in a similar way as for the first algorithm.

6 Final Remarks

The natural open problems left are to extend the structural results of this paper to 4-regular and 5-regular plane graphs and to find polynomial-time algorithms for recognizing the point sets that admit such graphs. As discussed before, it is also open whether there are point sets that admit a cubic plane graph but no connected cubic plane graph. Shortly before submitting the final version of this paper the authors have found a proof that no such point sets exist, i. e., if P admits a cubic plane graph it admits also a connected one.

References

[1] P. Bose, M. McAllister, and J. Snoeyink. Optimal algorithms to embed trees in a point set. *J. Graph Algorithms Appl.*, 1(2):1–15, 1997.

[2] T. K. Dey, M. B. Dillencourt, S. K. Ghosh, and J. M. Cahill. Triangulating with high connectivity. *Comput. Geom. Theory Appl.*, 8(1):39–56, 1997.

[3] A. García, F. Hurtado, C. Huemer, J. Tejel, and P. Valtr. On triconnected and cubic plane graphs on given point sets. *Comput. Geom. Theory Appl.*, 42:913–922, 2009.

[4] M. Gemignani. On finite subsets of the plane and simple closed polygonal paths. *Mathematics Magazine*, 39(1):38–41, 1966.

[5] C.-Y. Lo, J. Matoušek, and W. Steiger. Algorithms for ham-sandwich cuts. *Discrete & Computational Geometry*, 11:433–452, 1994.

[6] A. Tamura and Y. Tamura. Degree constrained embedding into points in the plane. *Inf. Process. Lett.*, 44:211–214, 1992.