Interval Stabbing Problems in Small Integer Ranges

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Outline

1. Problem Definitions

2. Data Structure

Interval Stabbing

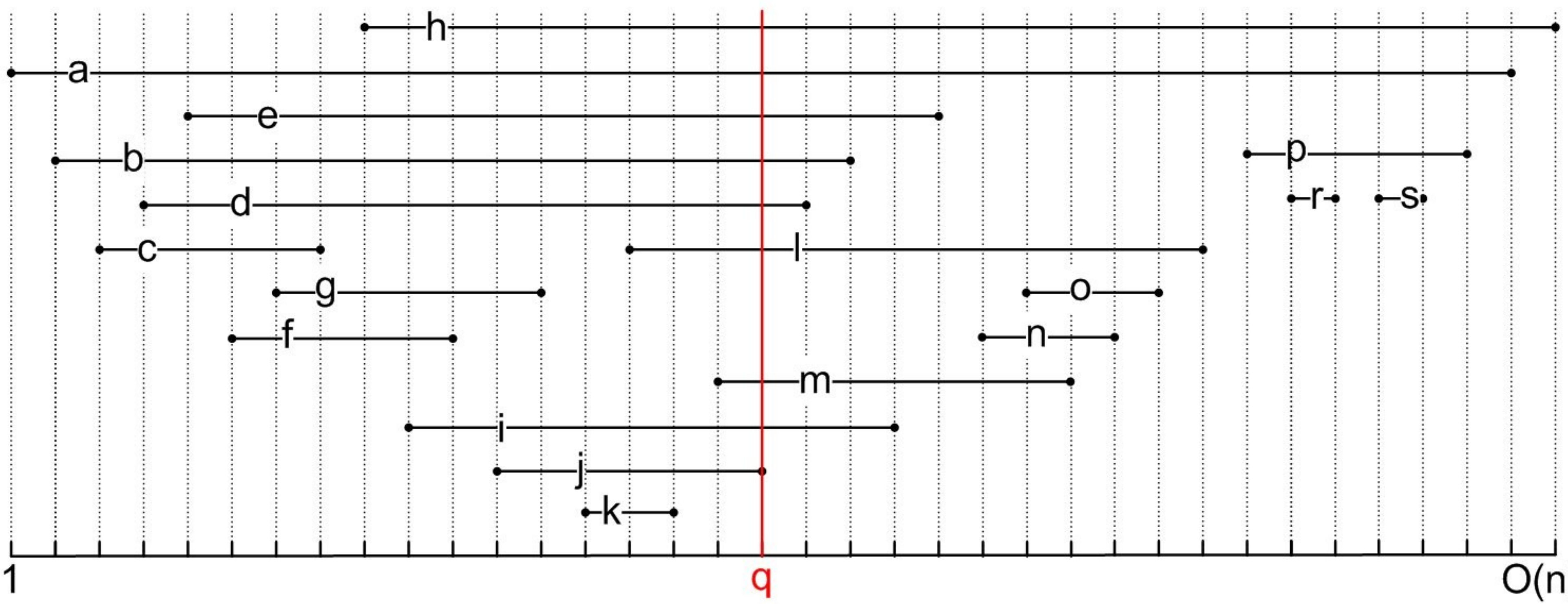
- ***I*** = set of *n* intervals $[l_i, r_i]$ with $l_i \leq r_i$

Stabbing query on a value *q:*

- Asks for all intervals in ***I*** that contain *q.*

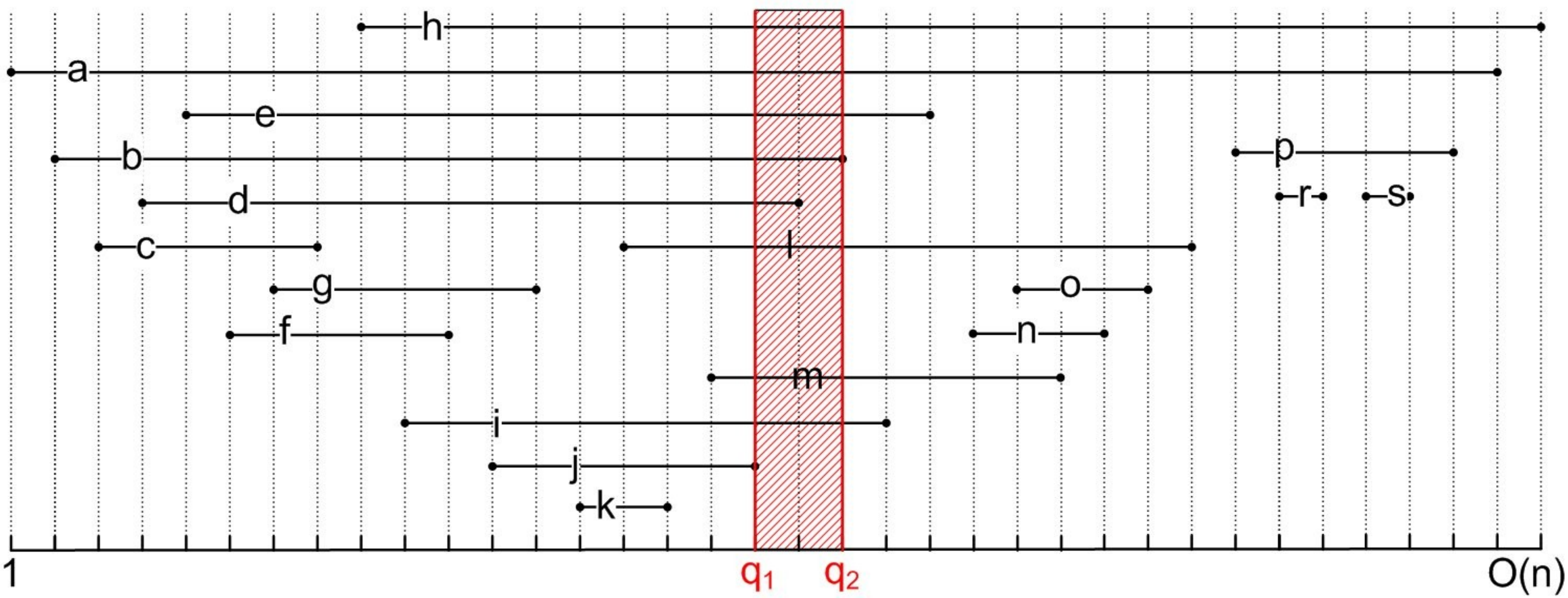
Wanted:

- Data structure that supports queries efficiently.



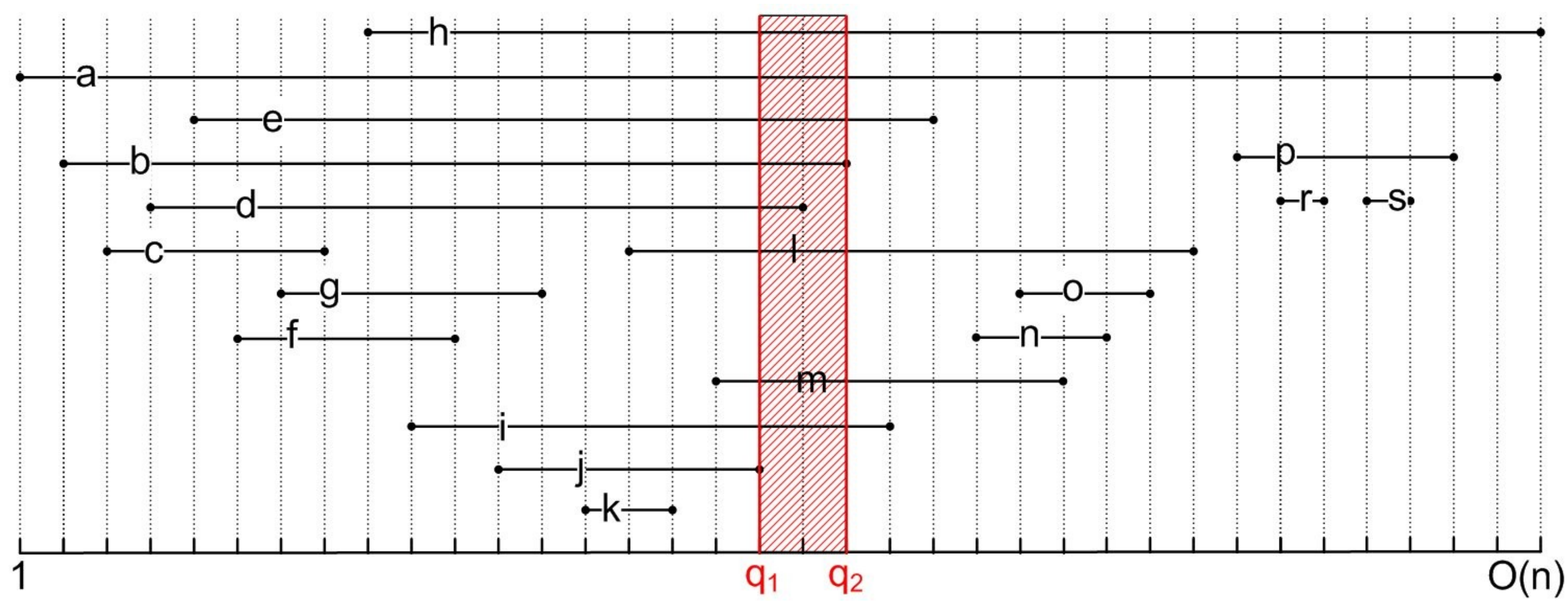
Interval Stabbing Problems

- **Interval Stabbing Problem**
- **Interval Intersection Problem:**
  - Given a query interval $[q_1,q_2]$, report all intervals in ***I*** that intersect $[q_1,q_2]$.



Interval Stabbing Problems

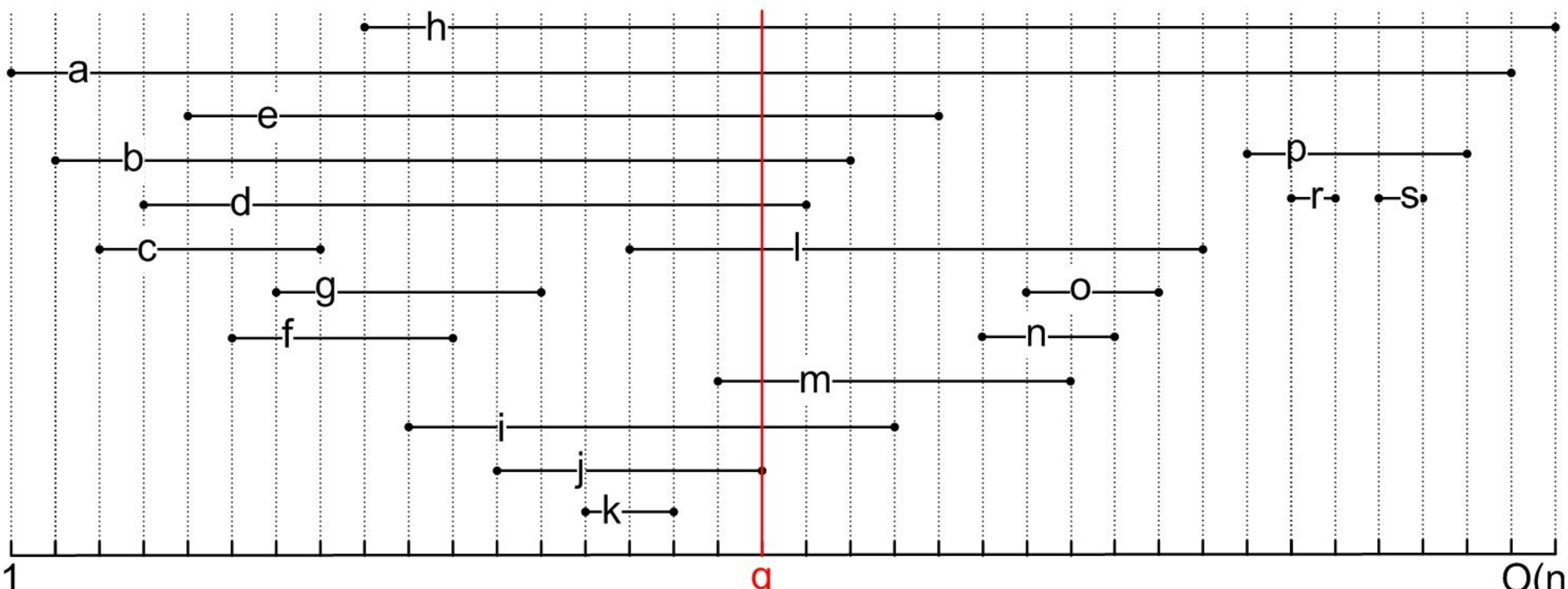
- **Interval Cover Problem:**
  - Given a query interval $[q_1,q_2]$ in I, report all intervals in I that contain $[q_1,q_2]$.
- **Multiple Query Problems:**
  - Given multiple queries sorted in lexicographic order, extend each prior problem to report intervals being contained in the union of outputs.



Interval Stabbing Problems

- Worst case running time for a query is *O(n).*
  ⇒ output-sensitive complexity

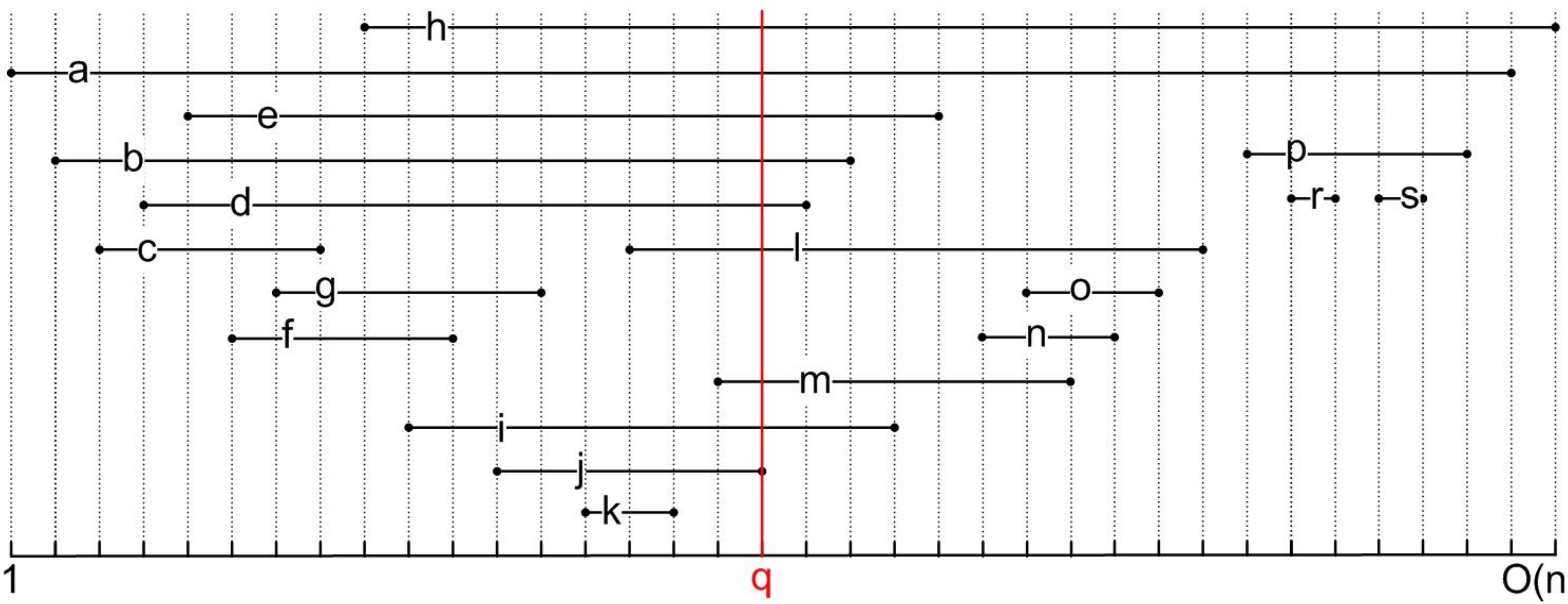
- We want optimal time *O(1+k)* for *k* intervals in the output.



Why Small Integer Ranges?

Let all interval endpoints be in *{1,...,N}*.

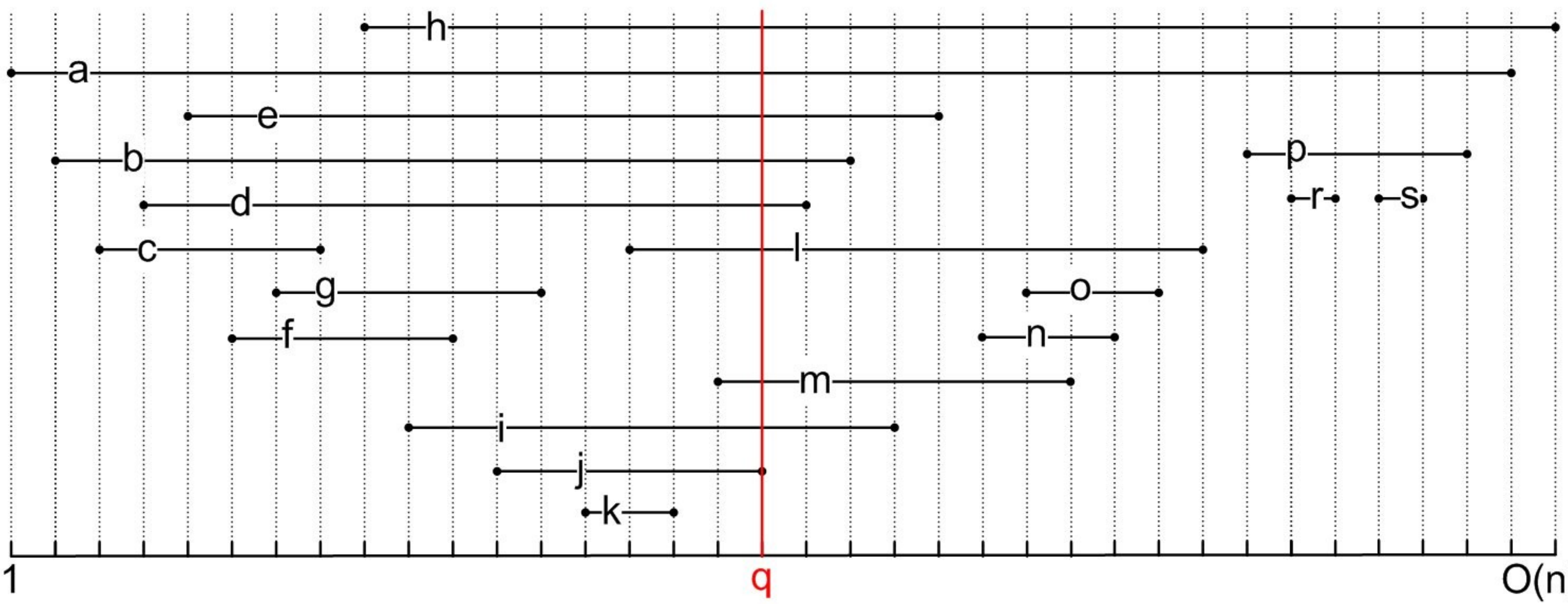
<u>Thm (Beame and Fitch 1999):</u>
For arbitrary *N*, <u>every data structure</u> using $n^{O(1)}$ memory cells needs $\Omega(\sqrt{\log(n)/\log(\log(n))})$ time for a stabbing query.



Why Small Integer Ranges?

To achieve constant time we have to impose a restriction:
⇒ We assume that all endpoints and q are in {1,...,O(n)}.

- W.l.o.g. all endpoints are pairwise distinct.



Overview

Wanted: Data structure for

- Interval Stabbing Problem
- Interval Intersection Problem
- Interval Cover Problem
- Multiple Query Problems

with

- *O(n)* preprocessing
- Stabbing queries in optimal time *O(1+k)*, output-sens.
- Output sorted by left endpoints

Overview

1986 Chazelle: Filtering Search

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We will focus on the first problem.

Outline

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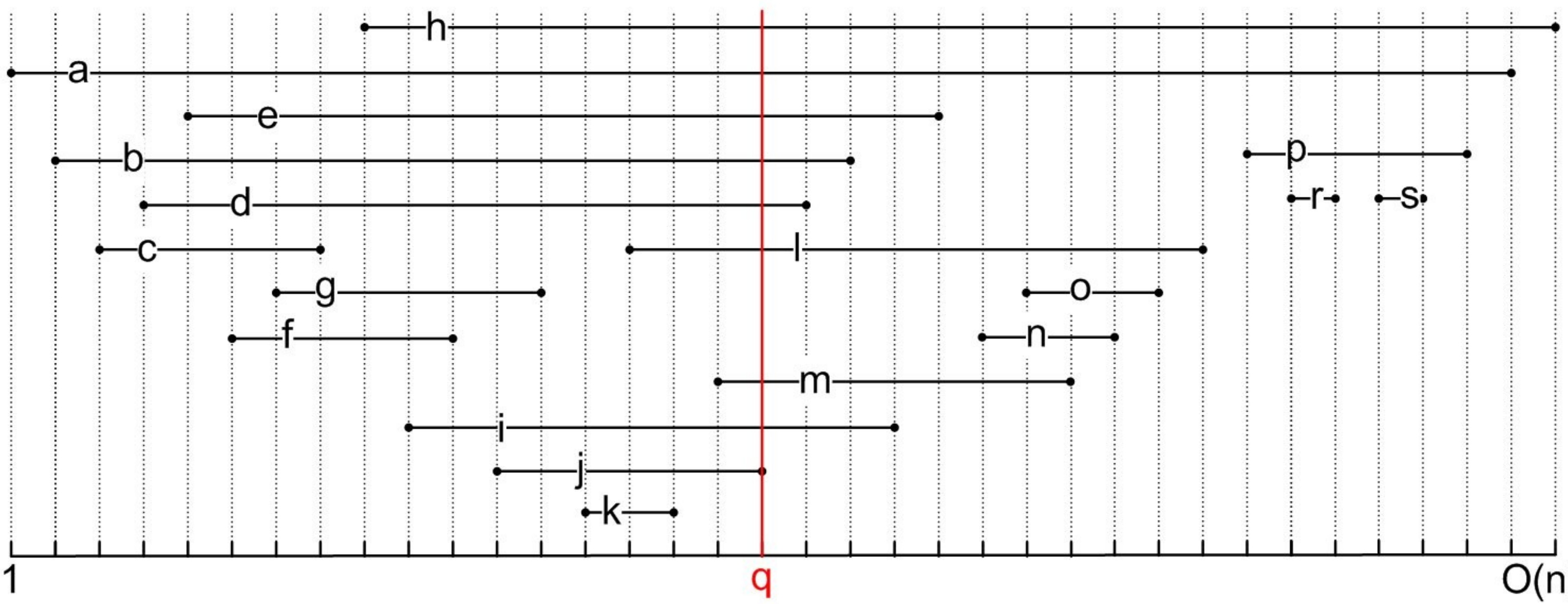
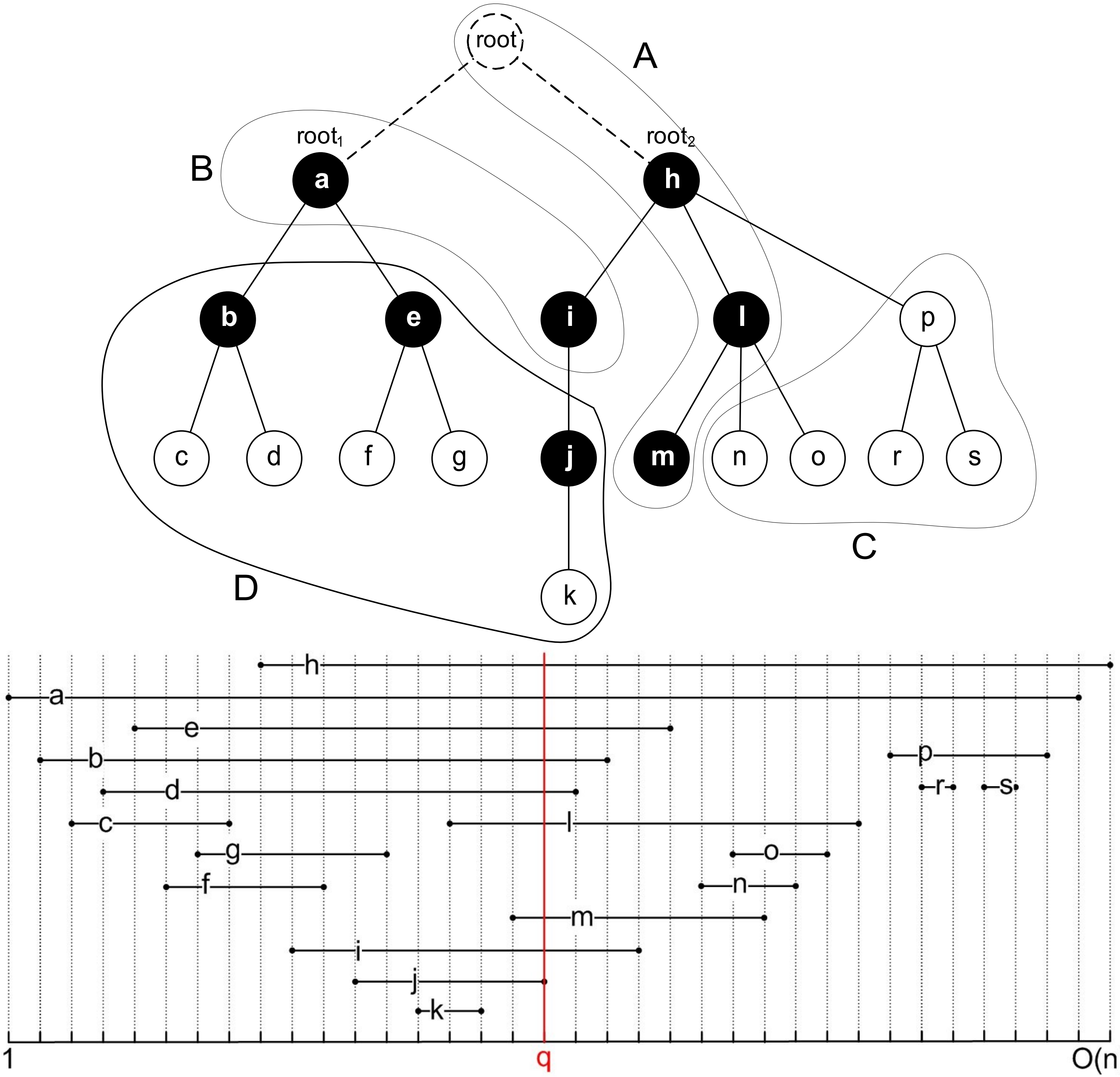
2. Data Structure

Data Structure

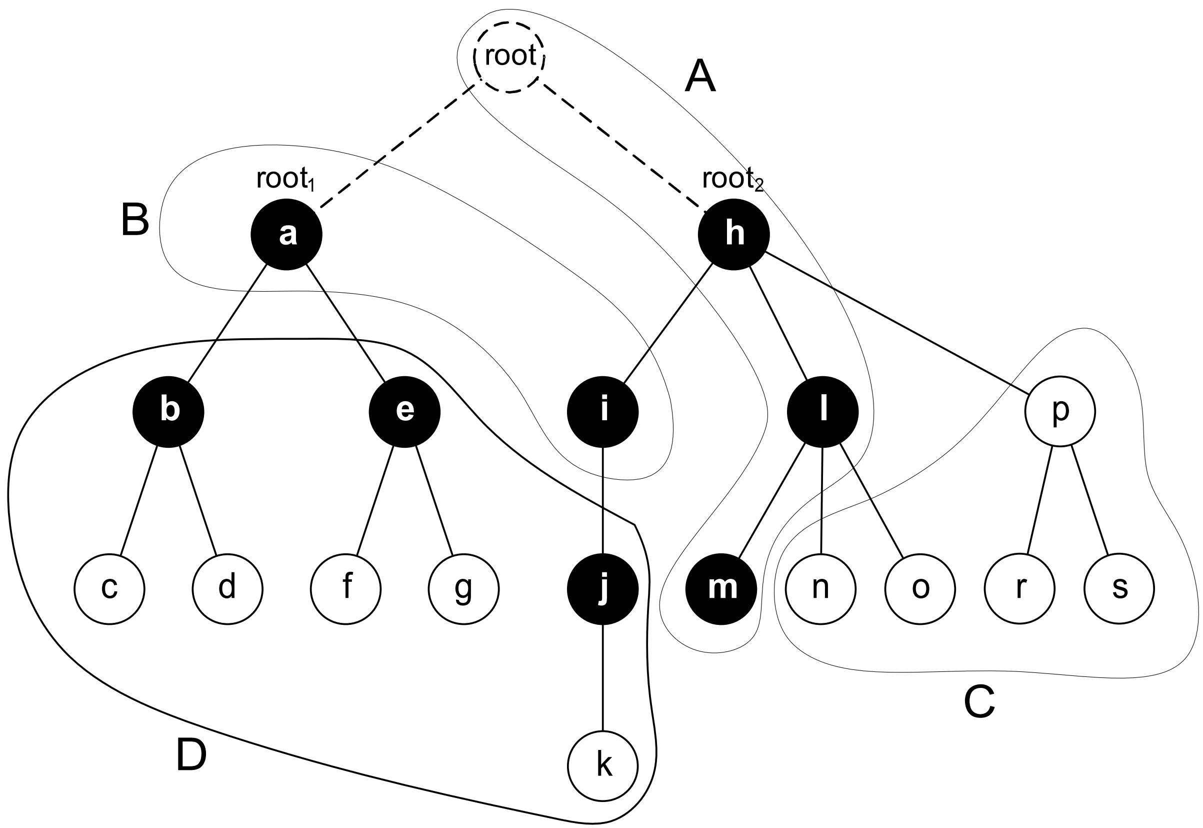
- An interval in a subset of *I* is *rightmost* if it is the one with maximum left endpoint.
- For an interval i:
  Parent(i) := rightmost interval that contains i



Data Structure



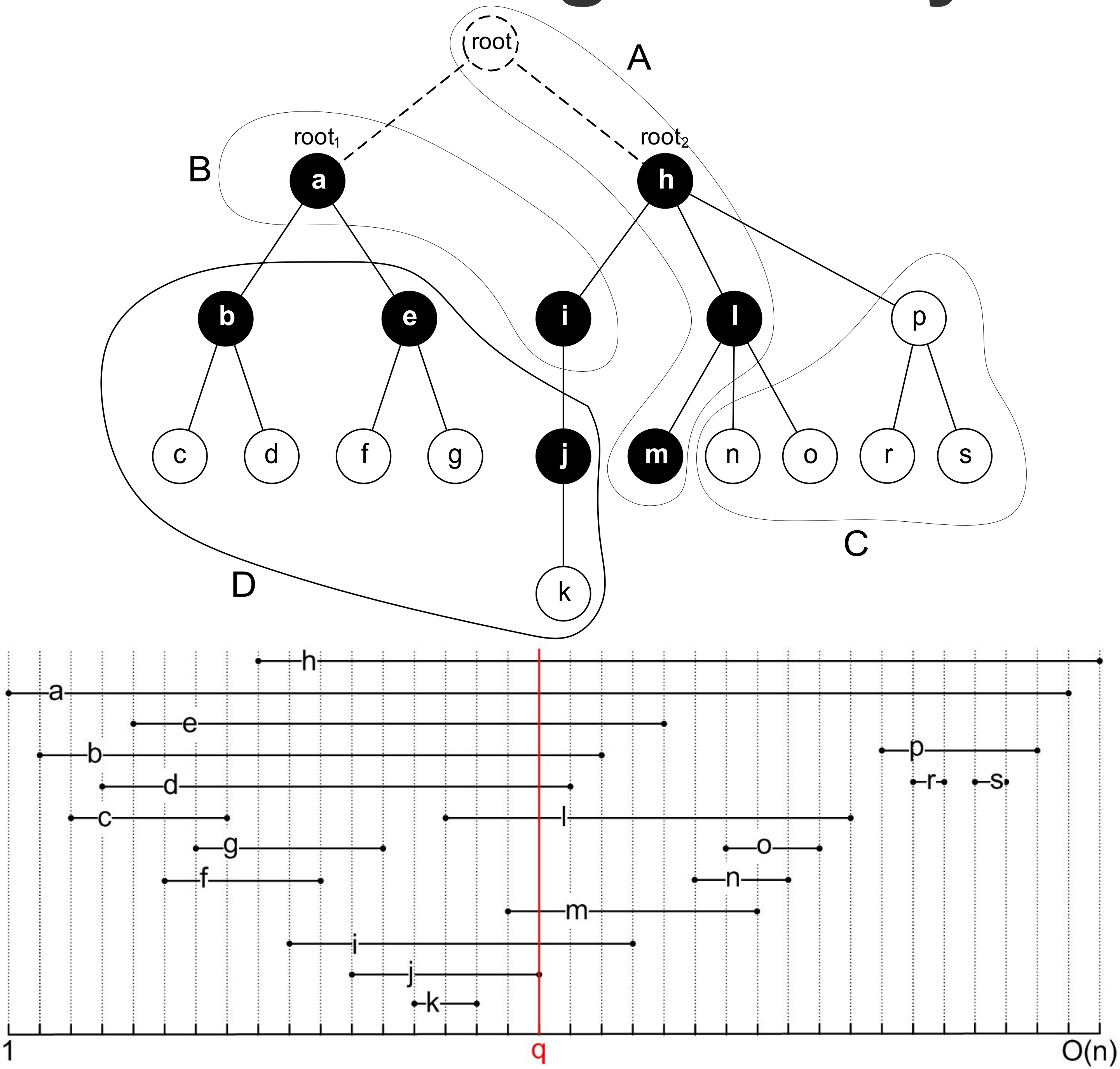
Data Structure



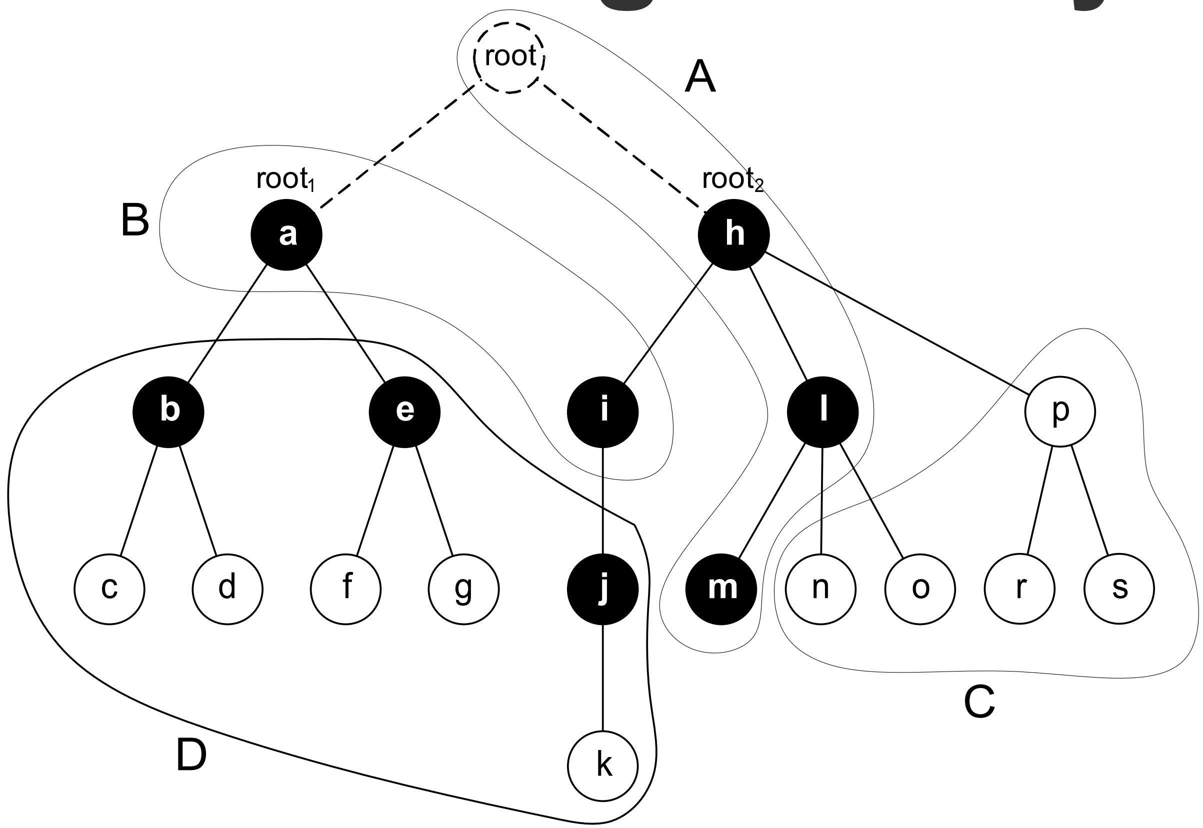
- All Parents can be computed in *O(n)* by a sweep line alg.
- Parents build a forest
- Data Structure: The forest + virtual root (trees ordered)

- We handle a query on *q* by traversing the forest from the (precomputed) rightmost interval containing *q*.

Processing a Query

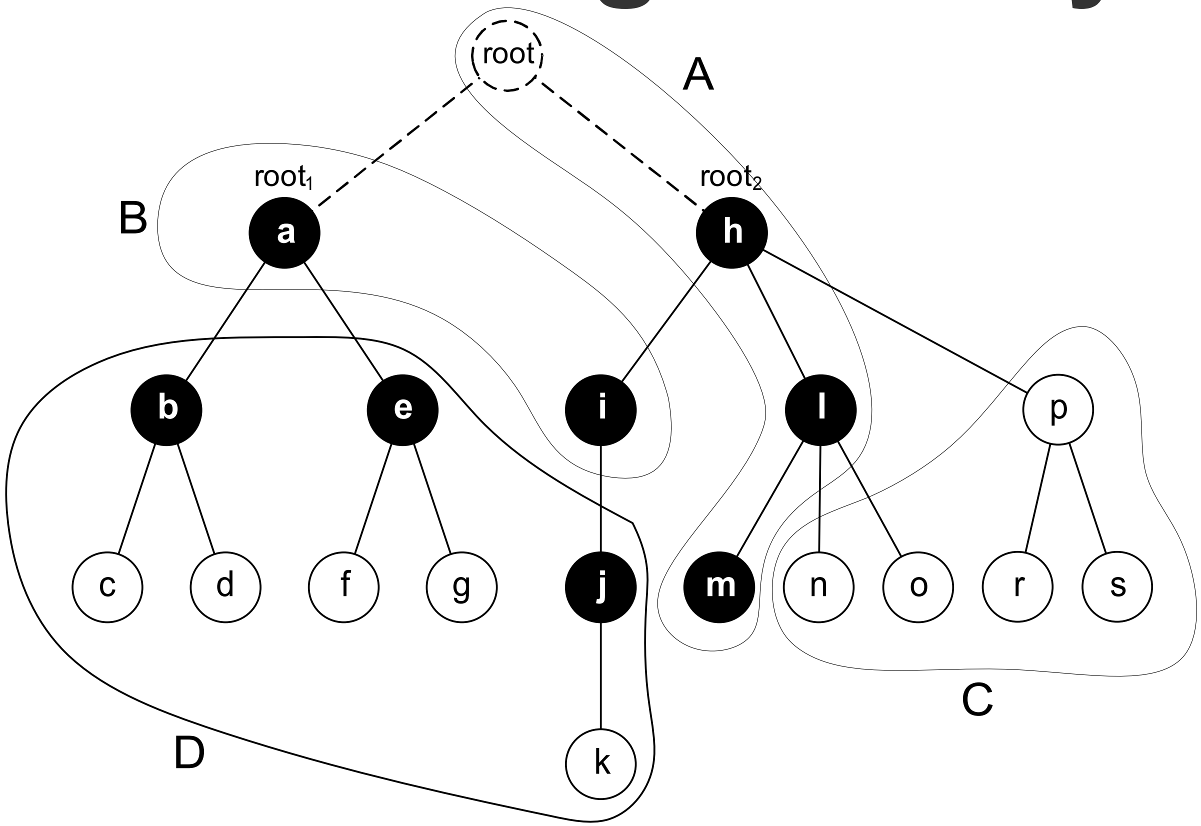


Processing a Query



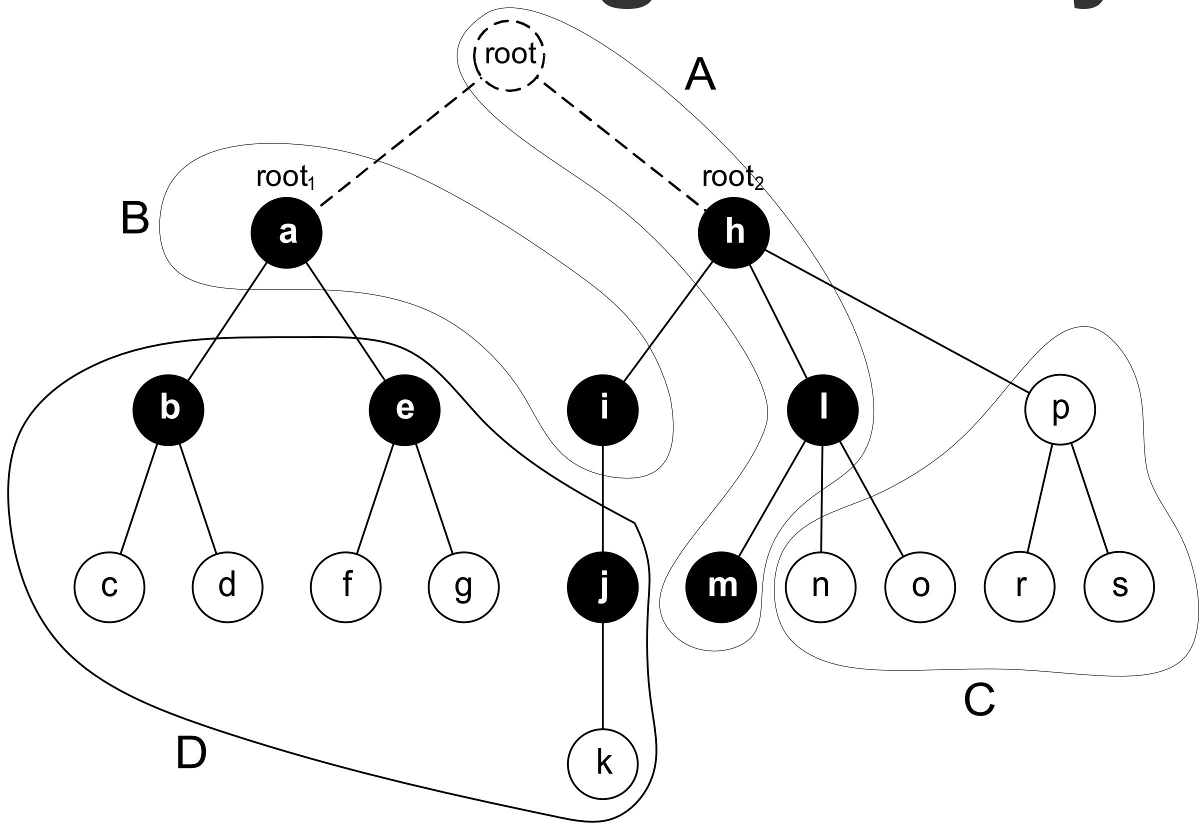
- All intervals in A are stabbed.
- No interval in C is stabbed.
- Stabbed siblings are adjacent.
  ⇒ Stabbed intervals in B can be computed efficiently.
- Only D remains.

Processing a Query



- Lemma 1: Every stabbed vertex $v \in D$ has a (stabbed) ancestor in *B*.

Processing a Query



- Lemma 2: The sibling *w* to the right of a stabbed vertex *v*∈*D* is stabbed as well, if it exists.

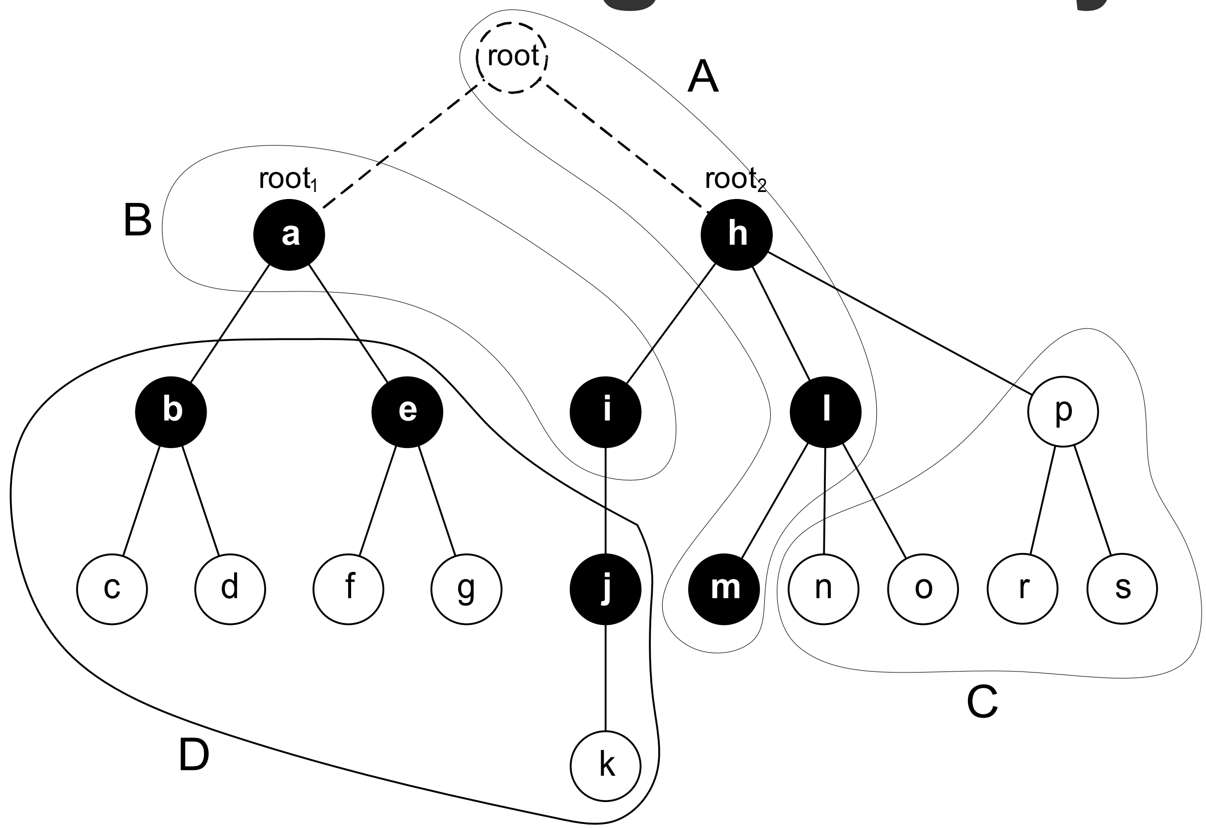
z'∈B∪A stabbed

z∈B stabbed

w∈D

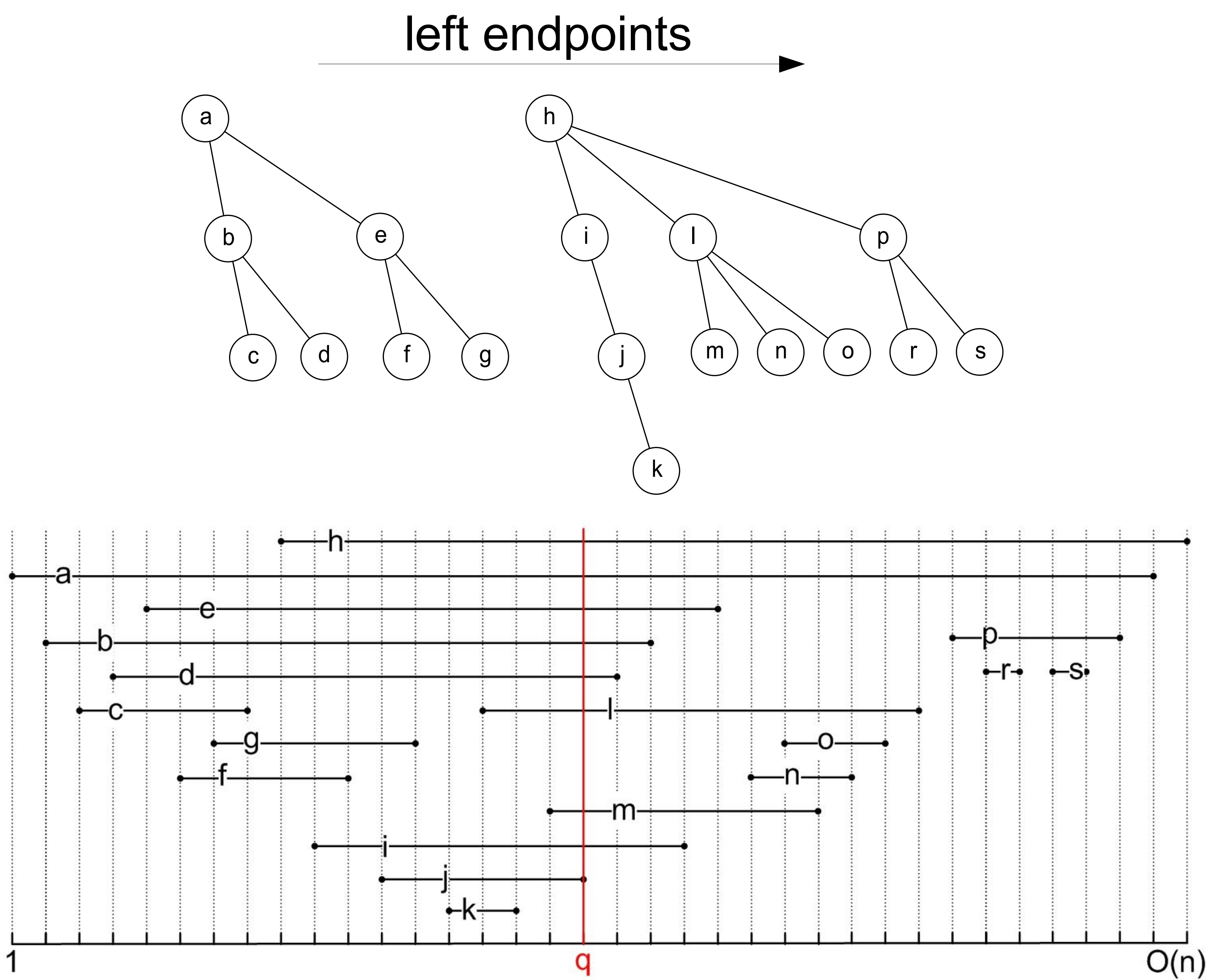
v∈D stabbed

Processing a Query



- It follows that every stabbed vertex $v \in D$ can be reached from a stabbed vertex in B by a zig-zag-path consisting of stabbed vertices.
- Only 3 directions to check on being stabbed:
  to the rightmost child, to the left, and up (only in A).

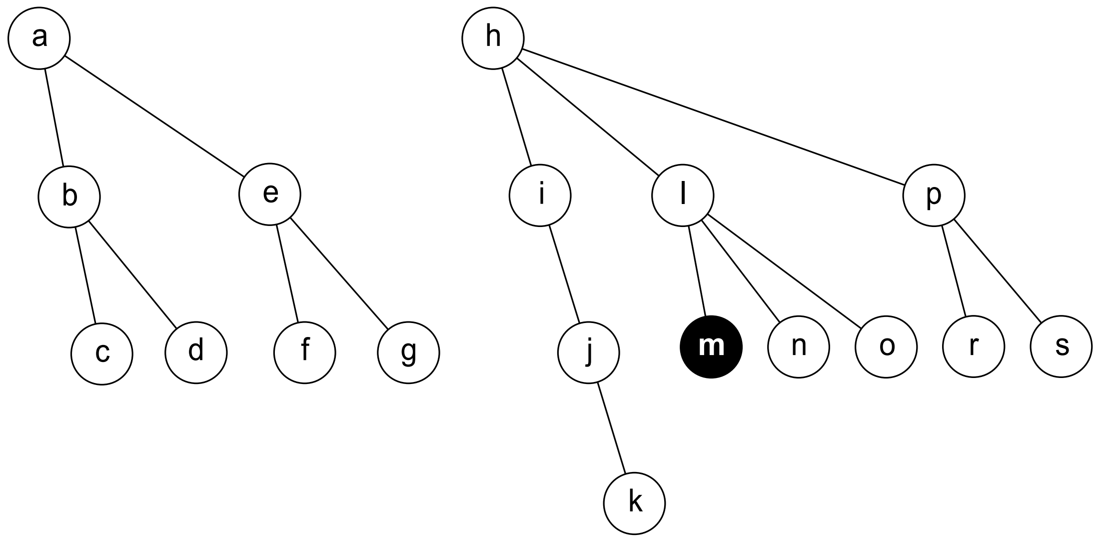
Example

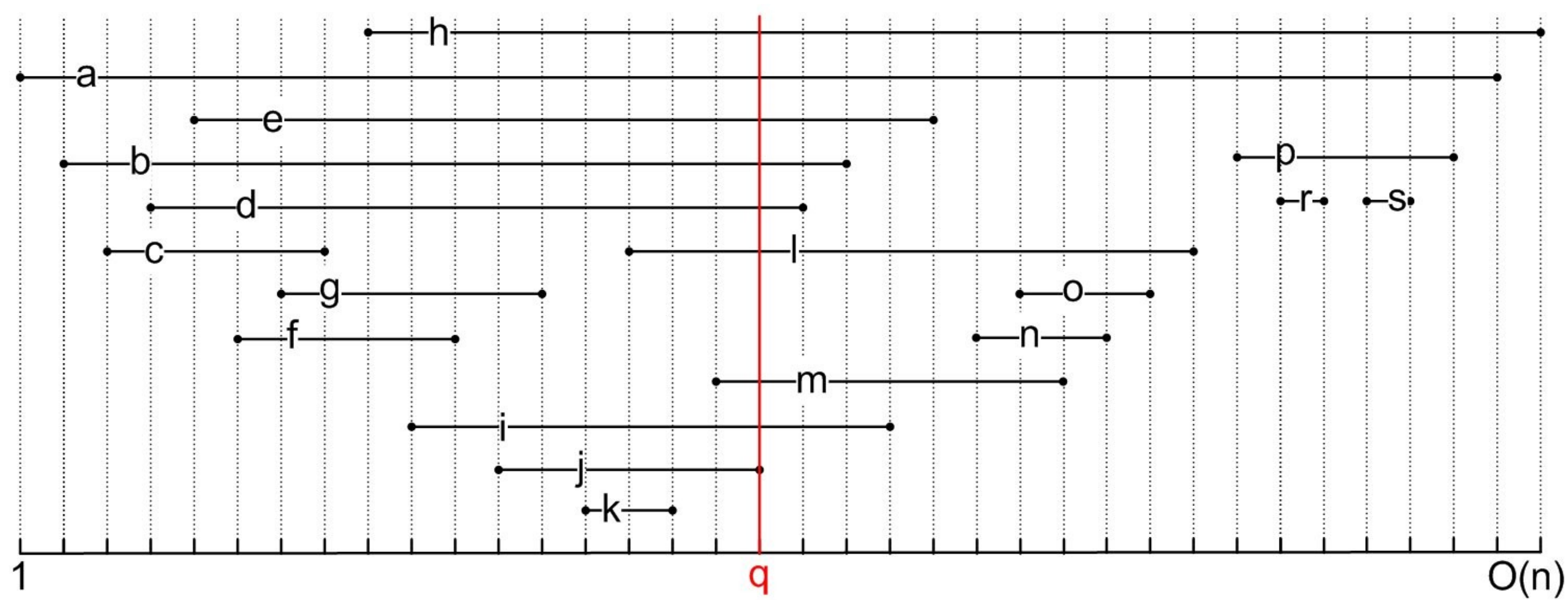


left endpoints
a
b
e
c
d
f
g
h
i
l
p
j
m
n
o
r
s
k
h
a
e
b
p
d
r
s
c
l
g
o
f
n
m
i
j
k
1
q
O(n)

Example

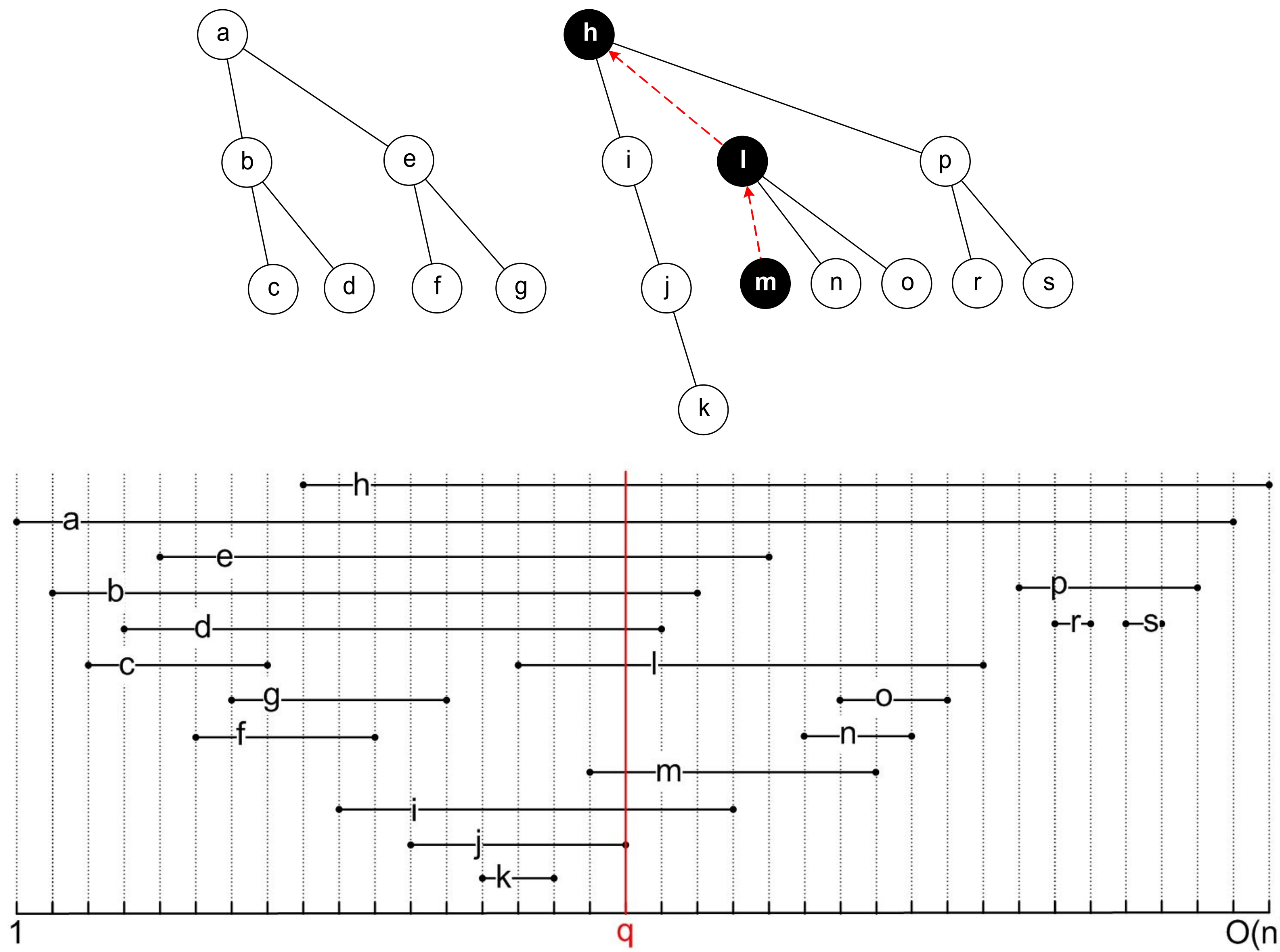
start traversal at the rightmost interval containing *q*





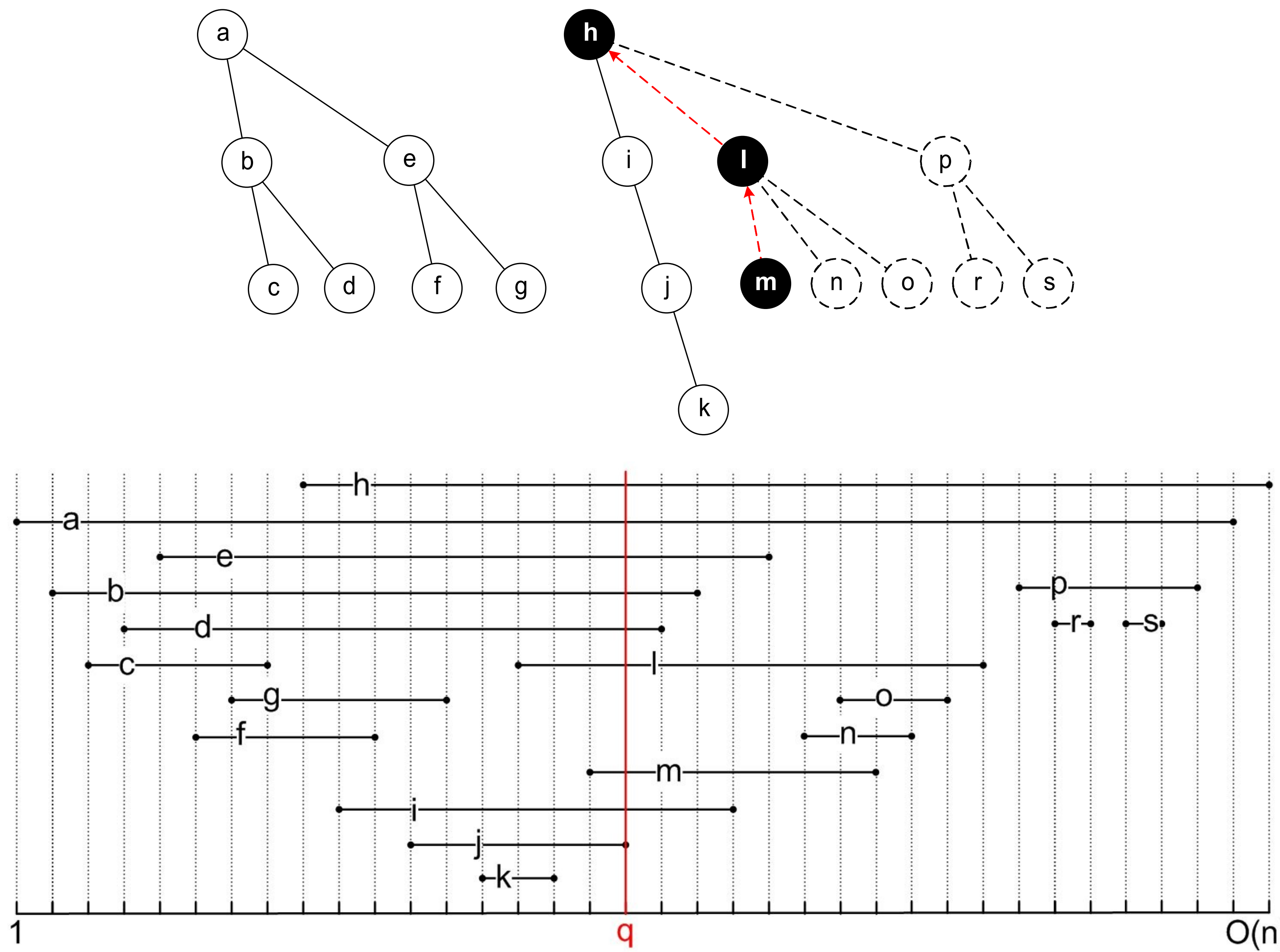
Example

ancestors of stabbed intervals are stabbed



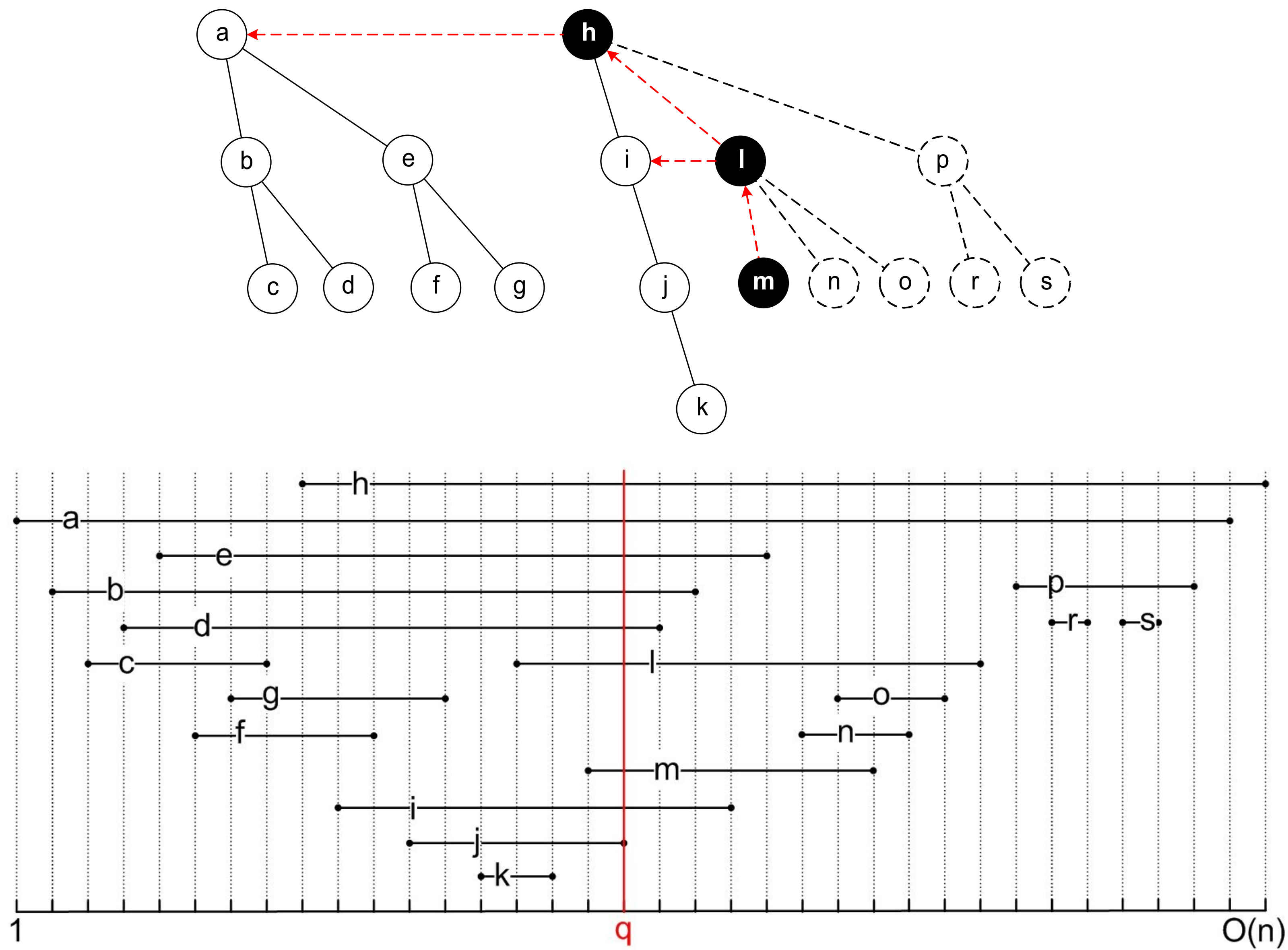
Example

C does not contain stabbed intervals



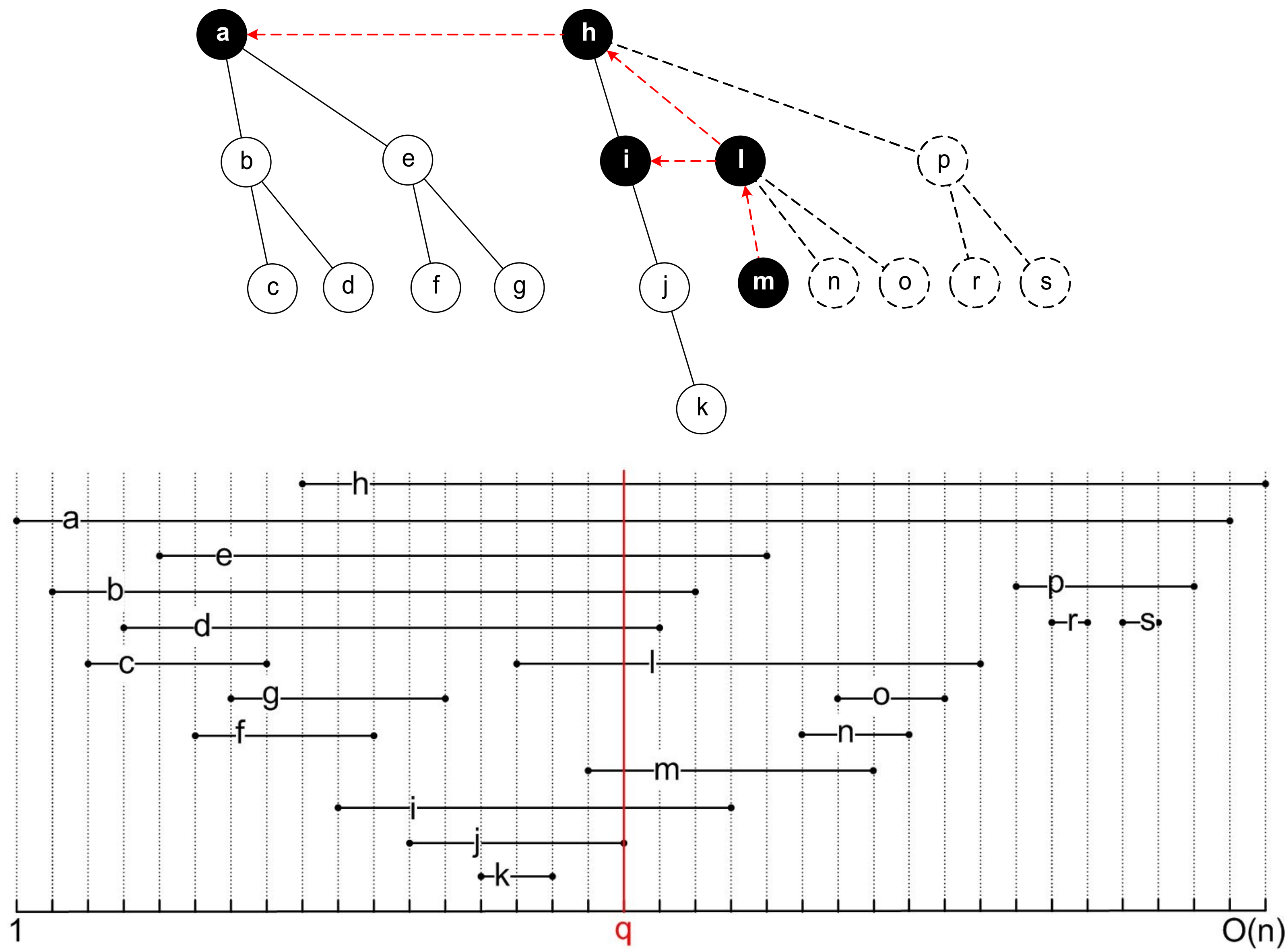
Example

check left siblings successively in *O(1)*...



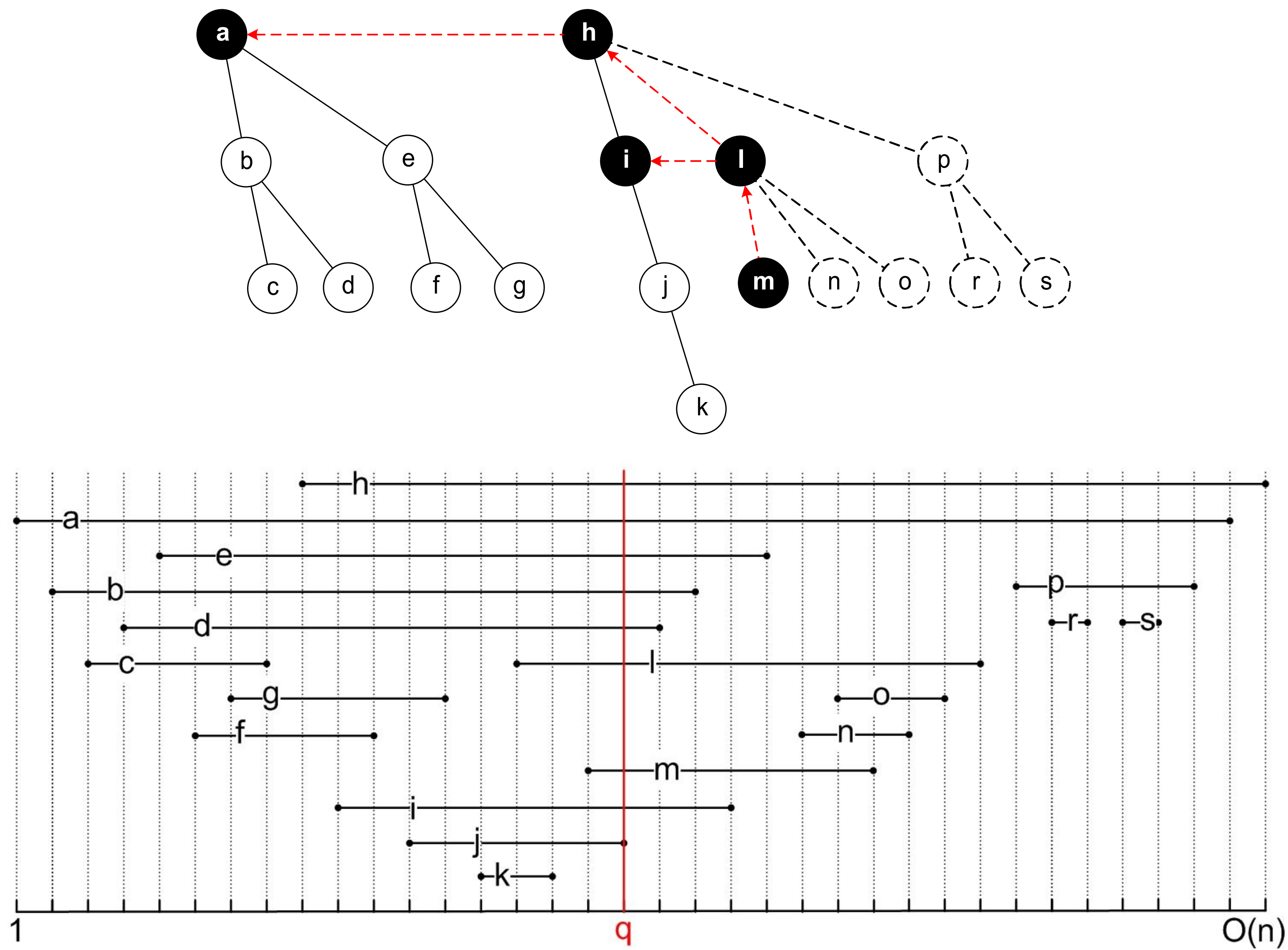
Example

...and get *B*



Example

try first to go to the rightmost child, then to go to the left



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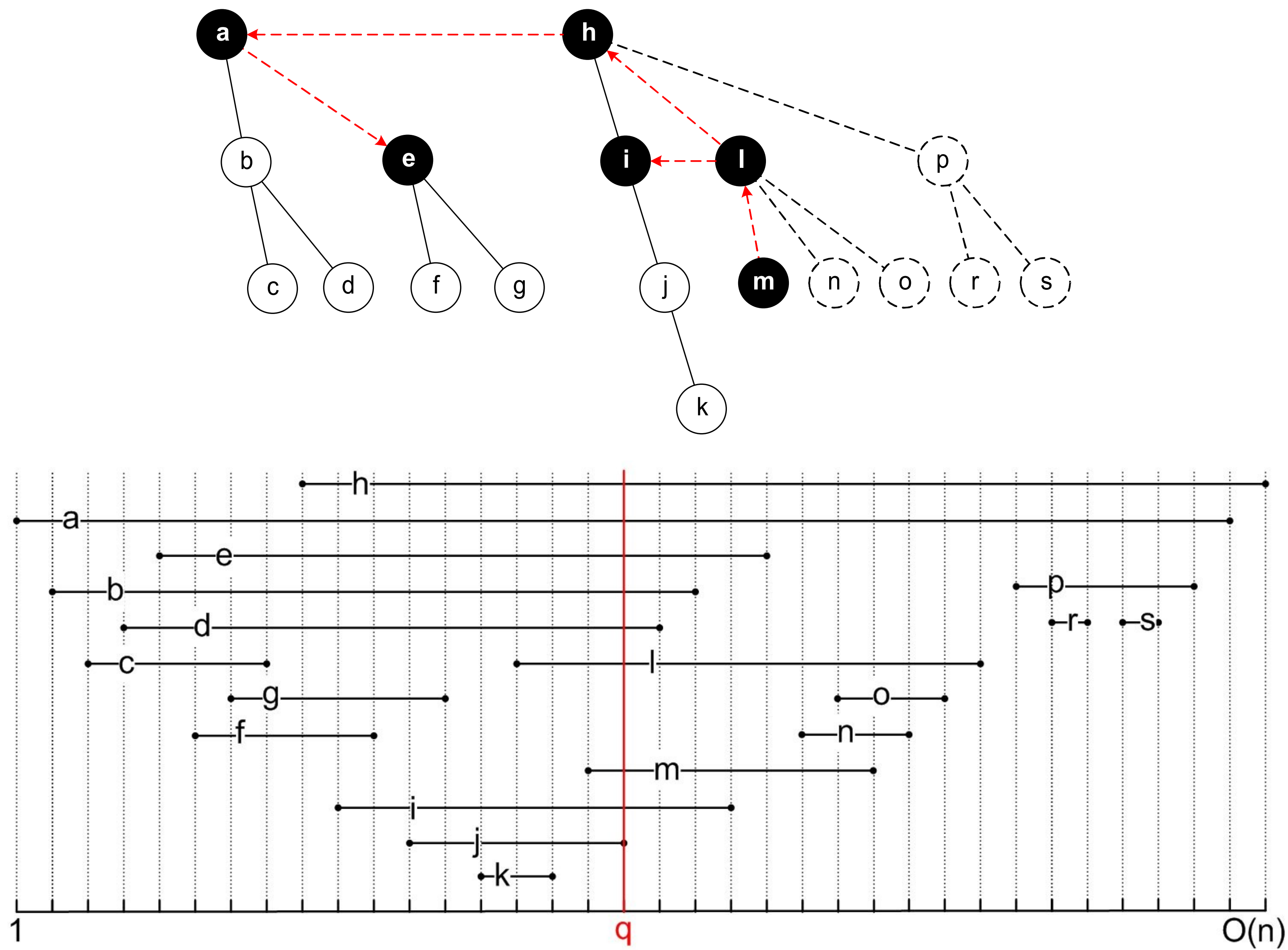

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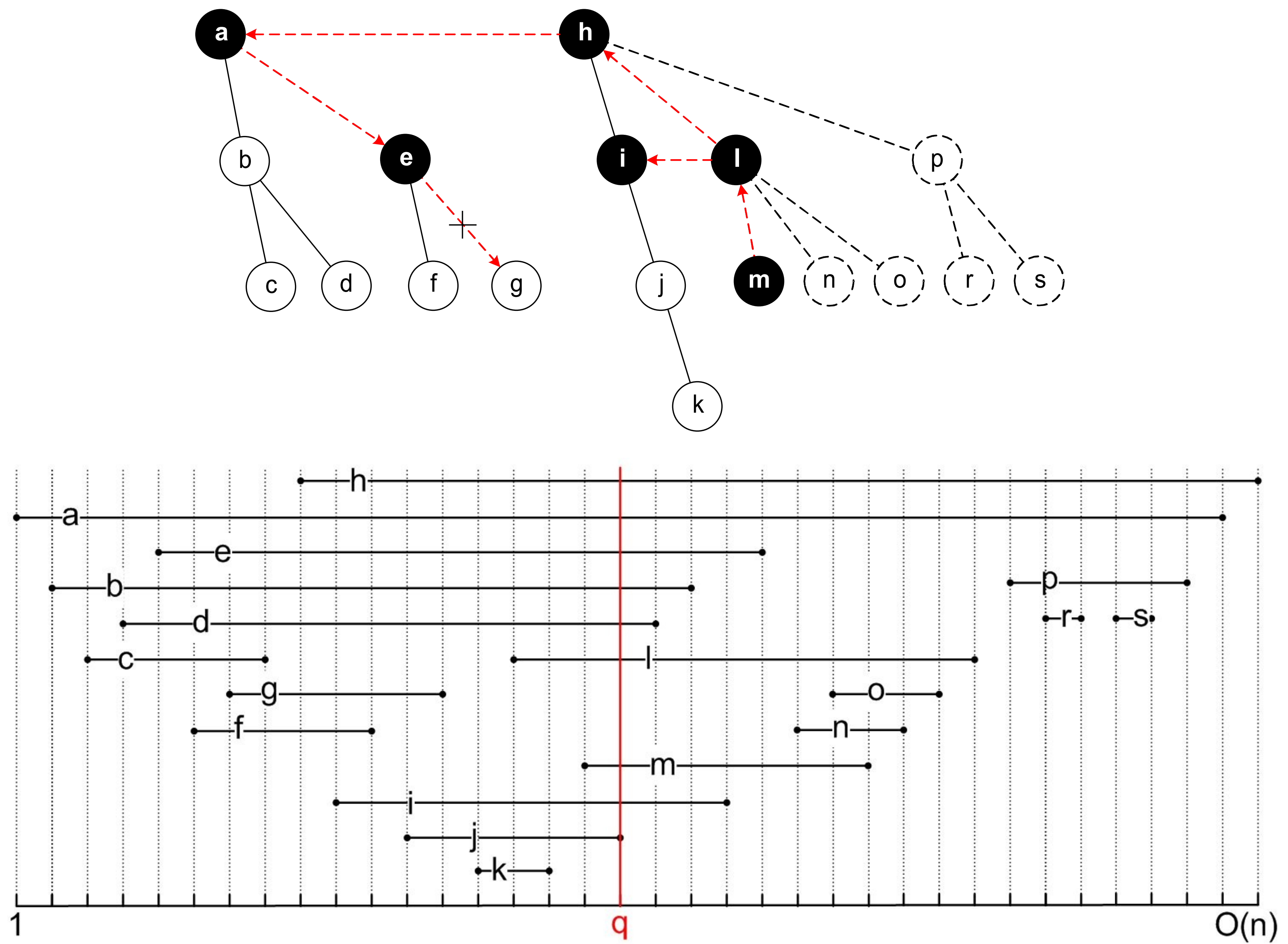

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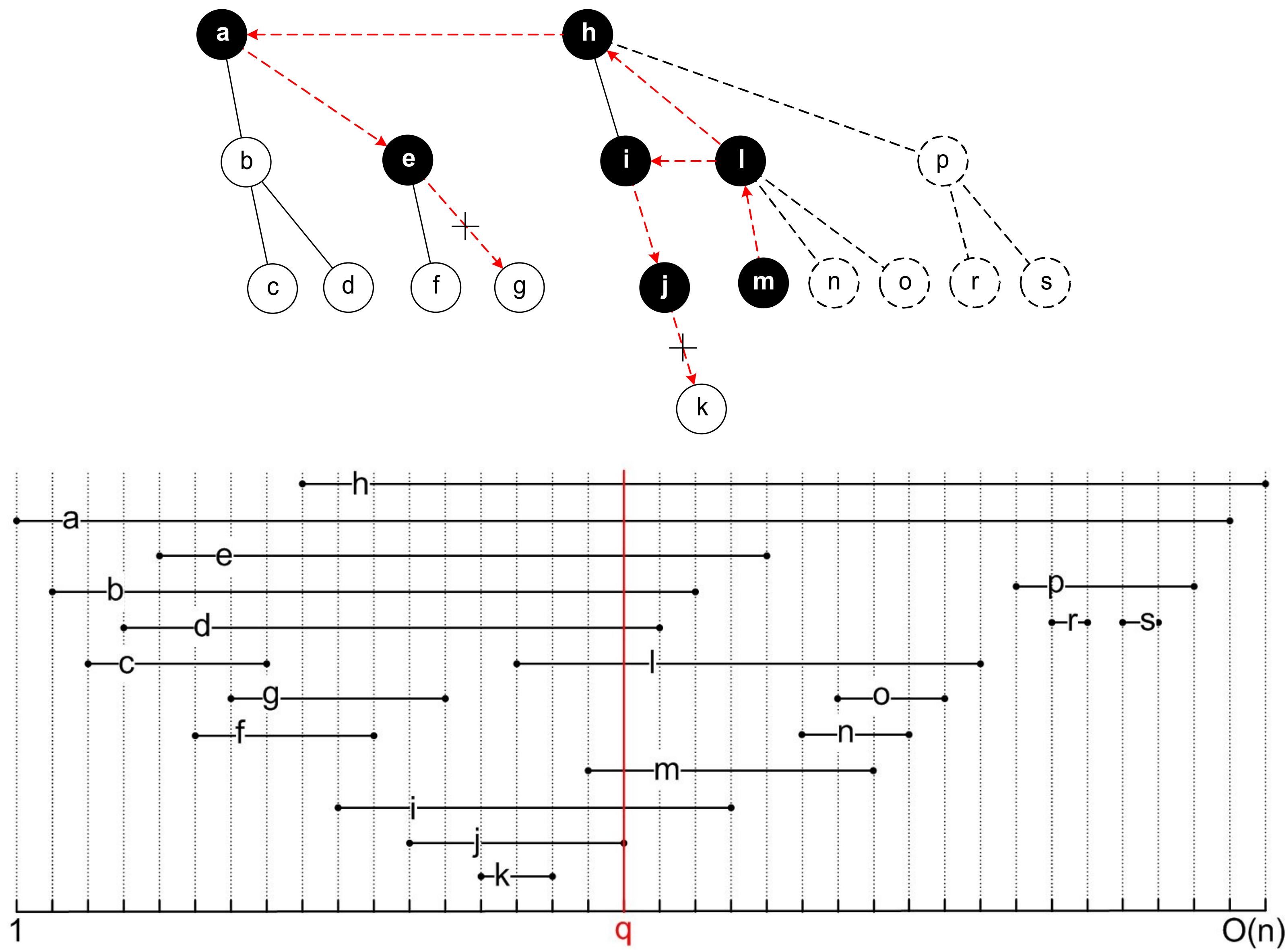

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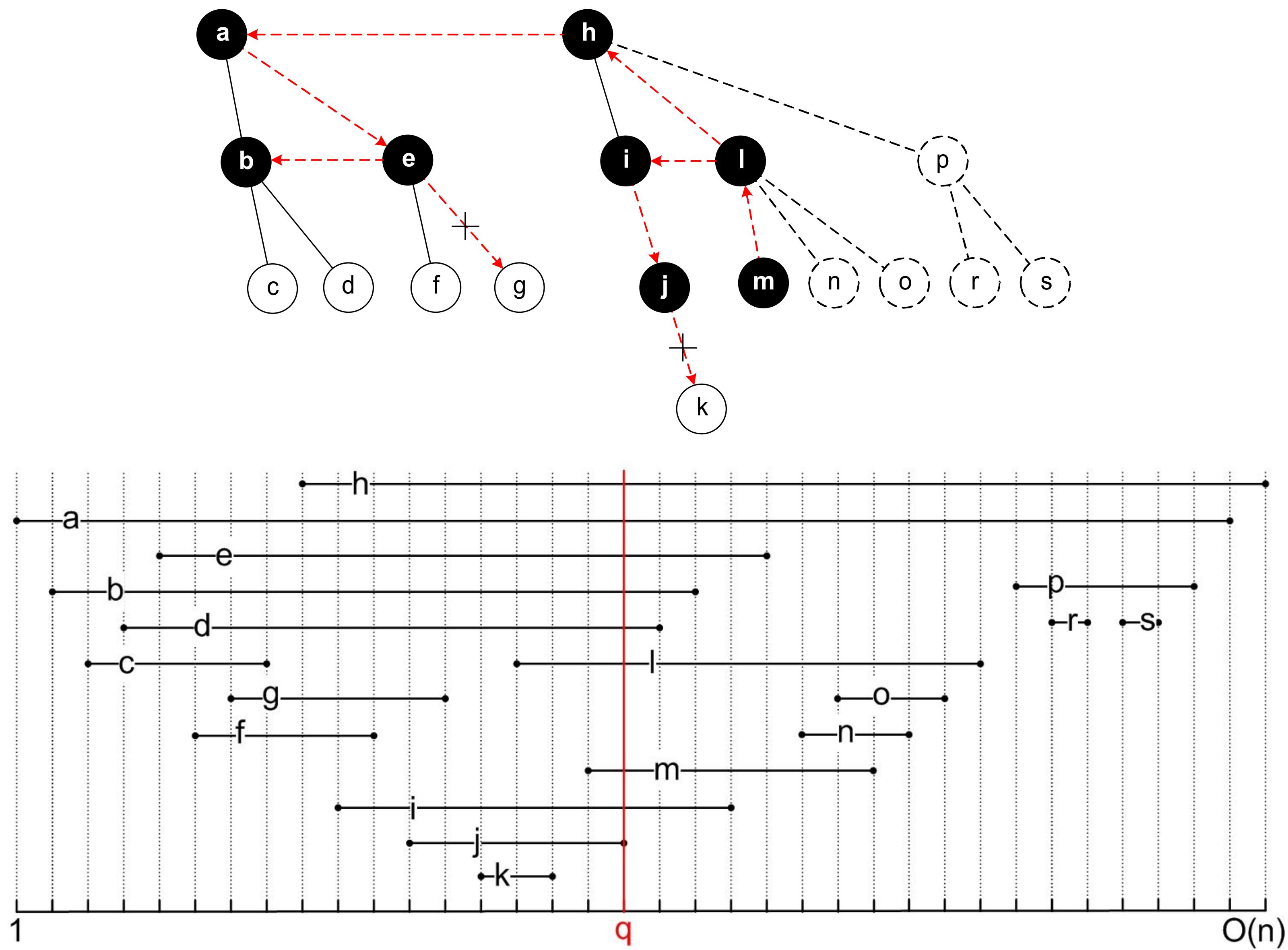

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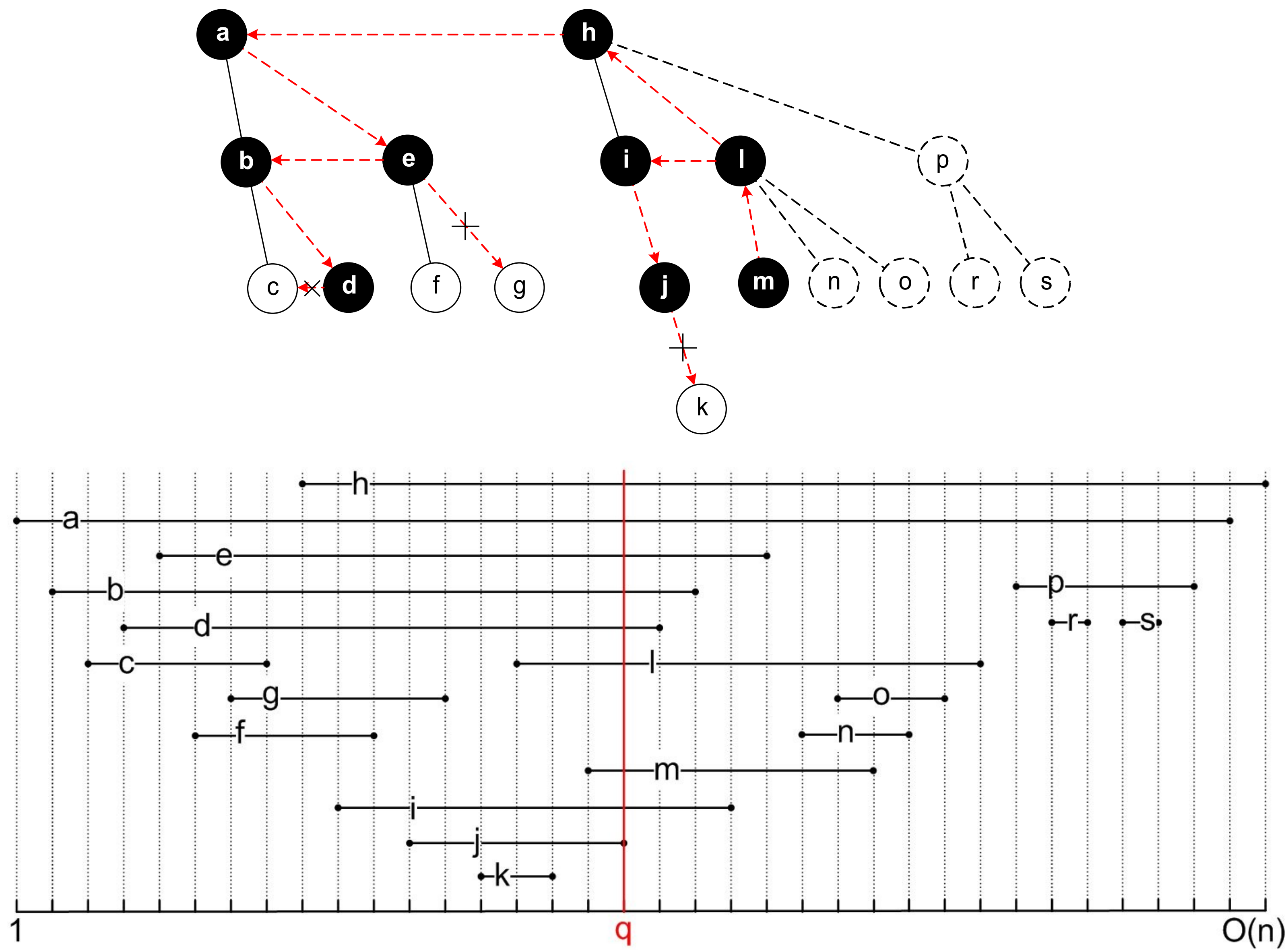

Problem Variants

- **Interval Intersection Problem:**
  - Perform query on the right endpoint of query interval, but change the stopping condition of the transversal.
- **Interval Cover Problem:**
  - Lemmas 1 and 2 still hold when *q* is an interval.
- **Multiple Query Problems:**
  - Start with the rightmost query and choose adaptively the next lower query value.