Minimal Schnyder Woods and Long Induced Paths in 3-Connected Planar Graphs*

Christian Ortlieb

Institute of Computer Science, University of Rostock, Germany

Abstract We investigate a new structural property of Schnyder woods: every minimal Schnyder wood of a 3-connected planar graph of order n has a tree of depth at least $\log_2(n)/(3\log_2(3))$. This bound is tight. Our result directly implies that such a graph has an induced path of length at least $\log_2(n)/(3\log_2(3))$, improving the previous best lower bound on the length of such a path.

Keywords: Induced path · 3-connected planar graph · Schnyder wood · depth

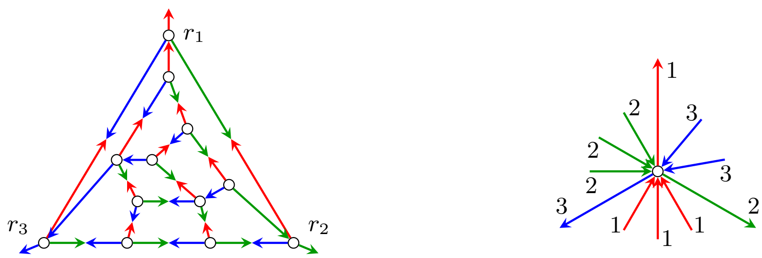
1 Introduction

Already in 1986, Erdős et al. [5] investigated the problem of finding long induced paths. Let $p(G)$ be the size, i.e., the number of vertices, of a longest induced path of G. For a connected graph G with radius $r(G)$, Erdős et al. [5] showed that $p(G) \geq 2r(G) - 1$. Fourteen years later, Arocha and Valencia [1] gave the lower bound $\log_{\Delta}(n)$ on the diameter (and hence on $p(G)$) for a 3-connected planar graph G of order n with bounded maximum degree Δ. For unbounded Δ, they gave an induced path of size $\sqrt{\log_3(\Delta)}$. In 2016, Di Giacomo et al. [4] showed that $p(G) \geq \frac{\log_2(n)}{12\log_2\log_2(n)}$ for 3-connected planar graphs G. And they gave an upper bound showing $p(G) \leq 1.3\log_2(n) + 5$ for a family of specific 3-connected planar graphs. The same year, Esperet et al. [6] improved the lower bound to $(\log_2(n) - 3\log_2\log_2(n))/6$ with a similar approach. Recently, we [15] improved the lower bound to $(1/6)\log_2(n)$ using a new technique based on deep trees in Schnyder woods.

In this paper, we give a better lower bound of $p(G) \geq \log_2(n)/(3\log_2(3)) \geq (1/4.76)\log_2(n)$. We approach the problem via deep trees in minimal Schnyder woods.

Given a planar embedding of a 3-connected planar graph and a minimal Schnyder wood on this embedding (Formal definitions are given in Section 2.), we show that at least one of the three trees has depth at least $\log_2(n)/(3\log_2(3))$. We also show that this bound is tight, i.e., for every $0 < \varepsilon < 1$ there exists a 3-connected planar graph with a minimal Schnyder wood such that every tree of

* This research is supported by the grant SCHM 3186/2-1 (401348462) from the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation).This preprint has not undergone peer review or any post-submission improvements or corrections. The Version of Record of this contribution is published in SOFSEM 2025: Theory and Practice of Computer Science and is available online at https://doi.org/10.1007/978-3-031-82697-9



(i) A Schnyder wood of the suspension of a 3-connected planar graph.

(ii) Example for Definition 1(iii) at a vertex in a Schnyder wood.

Figure 1: Illustrations for the definition of Schnyder woods.

this Schnyder wood has depth at most $\log_2(n)/(3(1-\varepsilon)\log_2(3))+1$. Actually, the 3-connected planar graph that we need for our lower bound has a unique Schnyder wood for our choice of the outer face. Thus, our bound is tight not only for minimal but also for arbitrary Schnyder woods of 3-connected plane graphs, that is, planar graphs with a fixed embedding. As mentioned above, the lower bound directly implies that $p(G) \geq \log_2(n)/(3\log_2(3))$. As stated in [15], this new structural property of Schnyder woods is not only of theoretical interest, but also comes with the following additional benefits.

We have an easy linear time algorithm that computes those long induced paths. Furthermore, we obtain that there are at least $f/(2\Delta)$ different such paths, where f is the number of faces and Δ the maximum degree. And, for every such path, there exists a planar grid drawing such that this path is monotone in both coordinates.

The paper is organized as follows. In Section 2, we give basic definitions and lemmas. In Section 3, we define the graph G^k and use it to give an upper bound on the depth of a tree in a minimal Schnyder wood. In Section 4, we give a procedure that transforms any 3-connected planar graph to G^k for a suitable k. This we use to give our lower bound. A section about dual Schnyder woods and the minimal Schnyder wood for 3-connected planar graphs, the omitted proofs and the proofs that are only sketched are provided in the appendix.

2 Schnyder Woods

We only consider simple undirected graphs. A graph is *plane* if it is planar and embedded into the Euclidean plane. Although parts of this paper use orientation on edges, we will always let vw denote the undirected edge $\{v,w\}$.

Let $\sigma := \{r_1, r_2, r_3\}$ be a set of three vertices of the outer face boundary of a plane graph G in clockwise order (but not necessarily consecutive). We call r_1, r_2 and r_3 *roots*. The *suspension* G^σ of G is the graph obtained from G by adding at each root of σ a half-edge pointing into the outer face. With a little abuse of notation, we define a *half-edge* as an arc that has a start vertex but no end vertex.

Definition 1 (Felsner [7]). *Let $\sigma = \{r_1, r_2, r_3\}$ and G^σ be the suspension of a 3-connected plane graph G. A* Schnyder wood *of G^σ is an orientation and coloring of the edges of G^σ (including the half-edges) with the colors 1,2,3 (red, green, blue) such that*

(i) Every edge e is oriented in one direction (we say e is unidirected*) or in two opposite directions (we say e is* bidirected*). Every direction of an edge is colored with one of the three colors 1,2,3 (we say an edge is i-*colored *if one of its directions has color i) such that the two colors i and j of every bidirected edge are distinct (we call such an edge i-j-*colored*). Throughout the paper, we assume modular arithmetic on the colors 1,2,3 in such a way that $i+1$ and $i-1$ for a color i are defined as $(i \mod 3)+1$ and $(i+1 \mod 3)+1$. For a vertex v, a uni- or bidirected edge is* incoming *(i-colored)* in *v if it has a direction (of color i) that is directed toward v, and* outgoing *(i-colored)* of *v if it has a direction (of color i) that is directed away from v.*

(ii) For every color i, the half-edge at r_i is unidirected, outgoing and i-colored.

(iii) Every vertex v has exactly one outgoing edge of every color. The outgoing 1-, 2-, 3-colored edges e_1, e_2, e_3 of v occur in clockwise order around v. For every color i, every incoming i-colored edge of v is contained in the clockwise sector around v from e_{i+1} to e_{i-1} (Figure 1ii). This clockwise sector includes e_{i+1} and e_{i-1}.

(iv) No inner face boundary contains a directed cycle in one color.

For an illustration of Definition 1 see Figure 1i.

For a Schnyder wood and color i, let T_i be the directed graph that is induced by the directed edges of color i. The following result justifies the name of Schnyder woods.

Lemma 1 ([9,18]). *For every color i of a Schnyder wood of G^σ, T_i is a directed spanning tree of G in which all edges are oriented to the root r_i.*

For a vertex v, we denote by $depth_i(v)$ the length of the v-r_i-path in the tree T_i. For a directed graph H, we denote by H^{-1} the graph obtained from H by reversing the direction of all its edges.

Lemma 2 (Felsner [8]). *For every $i \in \{1, 2, 3\}$, $T_i^{-1} \cup T_{i+1}^{-1} \cup T_{i+2}$ is acyclic.*

Using results on orientations with prescribed outdegrees on the respective completions, Felsner and Ossona de Mendez [9,14] showed that the set of Schnyder woods of a planar suspension G^σ forms a distributive lattice. The order relation of this lattice is defined on the superposition of the dual and the primal graph and also requires a Schnyder wood on the dual graph. We refer the interested reader to [17] for a definition of the minimal Schnyder wood for 3-connected planar graphs whose notation coincides with our notation.

But, as we are mostly working on planar triangulations, we give those definitions in the appendix and work with the following simpler statement. One can easily deduce from the result of Felsner and Ossona de Mendez [9,14] that for triangulations the order relation of this lattice relates a Schnyder wood of

G^σ to a second Schnyder wood if the former can be obtained from the latter by reversing the orientation of a directed clockwise cycle. This yields the following lemma.

Lemma 3 ([9, 14]). *Let G be a triangulated planar graph. The minimal element of the lattice of all Schnyder woods of G^σ contains no clockwise directed cycle.*

We call the minimal element of the lattice of all Schnyder woods of G^σ also the *minimal Schnyder wood* of G^σ. If the lattice has only one element, we say that the Schnyder wood is *unique*.

3 Upper Bound on the Maximum Depth of a Tree

In this section, we define a sequence of graphs with a minimal Schnyder wood such that for each $0 < \varepsilon < 1$ there exists an N such that in each of those graphs of order $n \geq N$ each tree of the Schnyder wood has depth at most $1/(3(1-\varepsilon)\log_2(3))\log_2(n) + 1$.

Those graphs are specific planar 3-trees. A *planar 3-tree* is a graph that can be constructed by the following procedure. Starting with a triangle, we iteratively select an internal face, add a new vertex v in its interior and connect this vertex with the three vertices of that face. During this construction, we assign a *level* to each newly added vertex v as follows. Every vertex on the outer face has level 0. And for v, we define $level(v) := \max\{level(w) \mid w \text{ is adjacent to } v\} + 1$. Observe that planar 3-trees are triangulated. Define the *complete* planar 3-tree of level k to be the 3-tree with the maximum number of vertices such that every vertex has level at most k. And let an *internal leaf* be a leaf of a tree of the Schnyder wood that is not on the boundary of the outer face. This construction procedure motivates the following lemma.

Lemma 4. *Let G be a triangulated plane graph and S be a Schnyder wood of G^σ. Let v be an internal leaf of the tree T_i for some $i \in \{1, 2, 3\}$. Let vp and vq be the outgoing $(i+1)$-colored edge and the outgoing $(i+2)$-colored edge at v, respectively. Let f be the internal face that has vp and vq on its boundary.*

If we add a vertex w in f and connect it to the vertices v, p and q, then there is exactly one way to augment S to a Schnyder wood of the suspension of $G + w$. If S is minimal or unique, then the resulting Schnyder wood is minimal or unique (w.r.t. the choice of the roots), respectively. Also, $depth_i(w) = depth_i(v) + 1$, $depth_{i+1}(w) = depth_{i+1}(v)$, $depth_{i+2}(w) = depth_{i+2}(v)$ and v is not a leaf of T_i in the resulting Schnyder wood.

For every $k \geq 1$, define G^k to be the planar 3-tree non-isomorphic to the triangle (as the triangle does not have internal leaves) such that in every tree of the Schnyder wood of its suspension every internal leaf has depth k. Observe that this is a valid definition. Let v be a leaf of w.l.o.g. T_3 in the Schnyder wood of the suspension of a planar 3-tree. Then, there exists a face that has v, its outgoing 1-colored edge vp and its outgoing 2-colored edge vq on its boundary. By Lemma 4, we can now add a vertex w in that face, connect it to p, q and v and color the new edges such that we obtain a Schnyder wood. In the resulting Schnyder wood, we have $depth_1(w) = depth_1(v)$, $depth_2(w) = depth_2(v)$, $depth_3(w) = depth_3(v) + 1$ and v is not a leaf in T_3 anymore. If we iterate this for every leaf of depth smaller than k, we eventually arrive at the graph G^k (Figure 2 and 3). Observe that the number of vertices in G^k rapidly increases. This in turn yields that the depth of the deepest tree in the Schnyder wood is small in terms of the number of vertices.

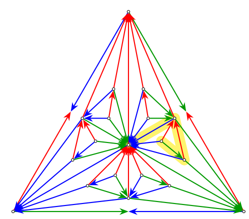


Figure 2: Illustration for the definition of G^k. G^2 together with its Schnyder wood. G^2 has 19 vertices. The path marked in yellow maps to the sequence $(3, 1, 2)$.

Lemma 5. *G^k has $3 + \sum_{s,t,r=0,\ldots,k-1} \binom{s+t+r}{s}\binom{t+r}{t}$ vertices. And for every $0 < \varepsilon < 1$ and $c > 0$ there exists a $K \geq 0$ such that $|V(G^k)| \geq c \cdot 3^{3(1-\varepsilon)(k-1)}$ for every $k \geq K$.*

Proof (Sketch). We give a bijection between the interior vertices of G^k and the sequences of the numbers 1, 2 and 3 in which each number appears at most $k-1$ times. Counting those sequences then shows the claimed statement.

By the definition of planar 3-trees, we have that every internal vertex v of G^k is adjacent to at least one vertex u such that $level(v) = level(u) + 1$. It is possible to show that for every internal vertex v except for the one vertex x that is adjacent to the three vertices on the outer face there exists exactly one such vertex u. Also, x is the only vertex of level 1. Hence, for every such vertex v, there is exactly one path $P_v = (v_0, \ldots, v_l)$ with $x = v_0$ and $v = v_l$ such that $level(v_s) = level(v_{s-1}) + 1$ for every $s = 1, \ldots, l$. This path maps to a sequence via $f(P_v) = (c(v_0v_1), \ldots, c(v_{l-1}v_l))$ where $c(e)$ refers to the color of the edge e (Figure 2). It is possible to show that f is a bijection between the interior vertices of G^k and the sequences of the numbers 1, 2 and 3 in which each number appears at most $k-1$ times.

Thus, we are left to count those sequences. We also count the vertices on the outer face, we use Stirling's formula and some $0 < \varepsilon < 1$ and obtain that

$$|V(G^k)| = 3 + \sum_{s,t,r=0,\ldots,k-1} \binom{s+t+r}{s}\binom{t+r}{t} \geq \frac{(3k-3)!}{((k-1)!)^3}$$
$$\geq exp\left(-\frac{96k-21}{432k^2-852k+420}\right) \cdot \frac{\sqrt{3}}{2\pi} \cdot \frac{1}{k-1} \cdot 3^{3\varepsilon(k-1)} \cdot 3^{3(1-\varepsilon)(k-1)}.$$

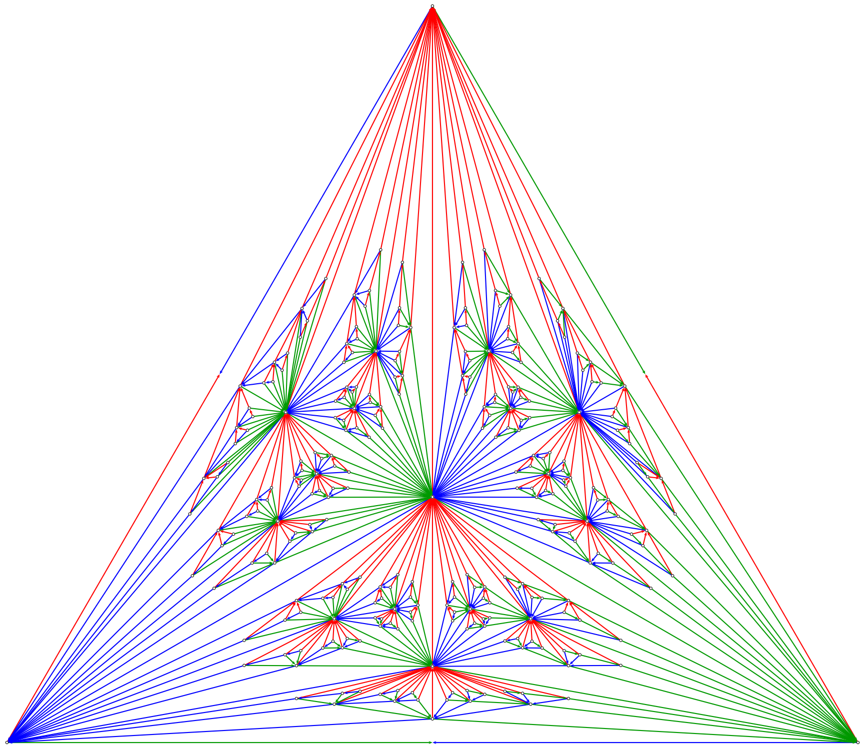


Figure 3: Illustration for the definition of G^k. G^3 together with its Schnyder wood. G^3 has 274 vertices.

Observe that

$$exp\left(-\frac{96k-21}{432k^2-852k+420}\right)\cdot\frac{\sqrt{3}}{2\pi}\cdot\frac{1}{k-1}\cdot 3^{3\varepsilon(k-1)}\longrightarrow\infty,\ \text{for } k\longrightarrow\infty.$$

Hence, for every $0<\varepsilon<1$ and $c>0$, there exists a $K\geq 0$ such that

$$exp\left(-\frac{96k-21}{432k^2-852k+420}\right)\cdot\frac{\sqrt{3}}{2\pi}\cdot\frac{1}{k-1}\cdot 3^{3\varepsilon(k-1)}\geq c$$

for every $k\geq K$. This concludes the proof.

Theorem 1. *For every $0<\varepsilon<1$ and n sufficiently large (n depends on ε), there exists a planar graph of order n with a unique, and thus, minimal Schnyder wood such that every tree of the Schnyder wood has depth at most $1/(3(1-\varepsilon)\log_2(3))\log_2(n)+1$. For ε small enough, we obtain $1/(3(1-\varepsilon)\log_2(3))\log_2(n)+1<1/4.75\log_2(n)+1$.*

Proof (Sketch). Remember that G^k and the Schnyder wood of its suspension are defined such that every tree of the Schnyder wood has depth k. It is possible to show that this Schnyder wood is also unique and thus minimal. By Lemma 5, for every $0 < \varepsilon < 1$ there exists a $K > 0$ such that $n := |V(G^k)| \geq 1 \cdot 3^{3(1-\varepsilon)(k-1)}$ for every $k \geq K$. Thus, we obtain

$$3^{3(1-\varepsilon)(k-1)} \leq n \Leftrightarrow k \leq \frac{1}{3(1-\varepsilon)\log_2(3)} \log_2(n) + 1.$$

Since $3\log_2(3) > 4.75$, we can choose ε such that $3(1-\varepsilon)\log_2(3) > 4.75$. Hence, $k \leq 1/4.75 \log_2(n) + 1$ for n sufficiently large.

4 Lower Bound

In this section, G always refers to a 3-connected plane graph such that S is the minimal Schnyder wood of G^σ. G might have additional structural properties if explicitly stated. We show that S has a tree of depth at least $1/(3\log_2(3)) \cdot \log_2(n)$. Define the depth of G to be the maximum depth of a tree of S and denote it by $depth(G)$. In our proof, we essentially show that G^k is indeed the worst case example. We give a procedure that transforms every given graph G to G^k for some $k \leq depth(G)$. Throughout this procedure, we only increase the number of vertices and decrease the depth of the deepest tree. In the end, we obtain that $depth(G) \geq depth(G^k) \geq 1/(3\log_2(3)) \cdot \log_2(|V(G^k)|) \geq 1/(3\log_2(3)) \cdot \log_2(|V(G)|)$. Hence, we need a lower bound on the depth of G^k.

Lemma 6. *We have that $n := |V(G^k)| \leq 1/2 \cdot 3^{3k-2} - 1/2$, and thus, $depth(G^k) = k > 1/(3\log_2(3)) \cdot \log_2(n)$.*

Proof. By Lemma 5, $n - 3$ equals the number of sequences of the colors 1, 2 and 3 such that each color appears at most $k-1$ times. This is clearly upper bounded by the number of sequences of length at most $3k-3$ such that the colors appear an arbitrary number of times. Hence, we obtain that

$$n - 3 \leq \sum_{l=0}^{3k-3} 3^l = \frac{3^{3k-2} - 1}{3 - 1} \Rightarrow \frac{1}{3\log_2(3)} \log_2(n) < k.$$

Lemma 7 (Di Battista et al. [3]). *The boundary of every internal face f of G can be partitioned into six paths $P_{1,3}$, $p_{2,3}$, $P_{2,1}$, $p_{3,1}$, $P_{3,2}$ and $p_{1,2}$ which appear in that clockwise order. For those paths the following holds (Figure 4).*

(i) $P_{i,j}$ consists of one edge which is either unidirected i-colored, unidirected j-colored or i-j-colored. Color i is directed in clockwise direction and color j in counterclockwise direction around f.

(ii) $p_{i,j}$ consists of a possibly empty sequence of i-j-colored edges such that color i is directed clockwise around f.

Lemma 8. *Let S be a minimal Schnyder wood and P be the counterclockwise 3-colored path on the boundary of some internal face. By Lemma 7, P consists of $p_{2,3}$ (a possibly empty sequence of 2-3-colored edges) and possibly $P_{1,3}$ (an edge which is either unidirected 1-colored, unidirected 3-colored or 1-3-colored). If $p_{2,3}$ is non-empty, then $P_{1,3}$ is either unidirected 3-colored or 3-1-colored.*

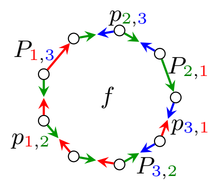


Figure 4: Illustration for Lemma 7. A face f and the paths on its boundary.

Lemma 8 allows for the following definition of $\tau(G)$. The definition is similar to the definition of $\tau(G)$ in [16]. The two graphs only differ on the outer face. A similar construction, but in the reverse direction, is used by Bonichon et al. [2].

Definition 2. *Define $\tau(G)$ to be a triangulation of G obtained as follows (Figure 5). First, add the edges r_1r_2, r_2r_3 and r_3r_1 if they are not yet in $E(G)$. Change the coloring of the edges incident to the roots such that the resulting orientation and coloring is still a Schnyder wood. If we add for example r_1r_2, then r_1r_2 is incoming 1-colored and outgoing 2-colored at r_1. And hence, the edge incident to r_1 that is 1-2-colored before we add r_1r_2 becomes unidirected 1-colored and incoming at r_1. Similarly, the edge incident to r_2 that is 1-2-colored before we add r_1r_2 becomes unidirected 2-colored and incoming at r_2.*

Let f be an internal face of G. Let P be the counterclockwise 3-colored path on the boundary of f and let $v_1, \dots, v_k$ be its vertices in counterclockwise order around f. If $k \geq 3$, proceed as follows. Add 3-colored edges $v_1v_k, \dots, v_{k-2}v_k$ directed towards v_k and for $j = 2, \dots, k-1$ change the color and orientation of v_jv_{j+1} such that v_jv_{j+1} is 2-colored and directed towards v_j. Proceed the same way for the counterclockwise 1-colored path and the counterclockwise 2-colored path on the boundary of f.

(i) An internal face of G. (ii) The corresponding subgraph of $\tau(G)$.

Figure 5: Illustration for the definition of $\tau(G)$. The counterclockwise 3-colored path P on the boundary of the face of G is highlighted in yellow.

Lemma 9. *The orientation and coloring S' of the suspension of $\tau(G)$ we obtain by Definition 2 is a minimal Schnyder wood. And we have $depth(\tau(G)) \leq depth(G)$.*

For the subsequent proofs to work, we need that every internal leaf has the same depth. Hence, we give the following definition. By Lemma 4, this can easily be achieved.

Definition 3. *Let G be triangulated and of depth k. Define $\overline{G}$ to be the graph obtained from G by the following iterative process. For all $i \in \{1, 2, 3\}$, whenever there is an internal leaf v in T_i that does not have depth k, we add a vertex u in the face delimited by the outgoing edge vp of v in color $i+1$, the outgoing edge vq of v in color $i+2$ and pq. We orient and color the edges incident to u such that we obtain a Schnyder wood, i.e., the edges uv, up and uq are outgoing at u and i-, $(i+1)$- and $(i+2)$-colored, respectively.*

Remark 1. Observe that in the setting of Definition 3, we obtain the following. By Lemma 4, $depth_i(u) = depth_i(v) + 1$, $depth_{i+1}(u) = depth_{i+1}(v)$, $depth_{i+2}(u) = depth_{i+2}(v)$ and v is not a leaf of T_i anymore. Hence, in $\overline{G}$, every internal leaf has depth k. Also, by Lemma 4, the resulting Schnyder wood is still minimal.

Lemma 10. *Let L_1 be the set of internal leaves of T_1 in $\overline{G}$. Let C be a facial cycle, i.e., a cycle that forms the boundary of a face, of $\overline{G} - L_1$. Then, there is no 3-colored edge e in $\overline{G}$ with head w in the interior of C and tail v on C. Symmetrically, this holds for the colors 2 and 3.*

Lemma 11 (folklore). *Let G be a triangulated plane graph of order n. Denote by l_i° the number of internal leaves of the tree T_i and by f° the number of internal faces of G. Then, $\sum_{i=1}^{3} l_i^\circ \leq f^\circ = 2n - 5$.*

Lemma 10 allows for the following definition.

Definition 4. *Let G be of depth k. Define G_1 to be the graph obtained by the following process. First, triangulate G as described in Definition 2 obtaining $\tau(G)$. Then, add vertices as described in Definition 3 obtaining $\overline{\tau(G)}$.*

Let L_1 be the set of internal leaves of $\overline{\tau(G)}$ of the tree T_1. Let $G_1 = \overline{\tau(G)} - L_1$. Now, for every facial cycle C of G_1 that is not a triangle do the following. For every vertex z on C with an outgoing 2-colored edge zy such that $zy \in E(\overline{\tau(G)})$ and $y \in L_1$ is in the interior of C, let v_z be the vertex where the 2-colored path in $\overline{\tau(G)}$ from z to the root r_2 first meets C. Add the edge zv_z, color it with color 2 and orient it from z to v_z. In order to guarantee that all internal leaves of T_2 and T_3 in G_1 have depth k and all internal leaves of T_1 have depth $k-1$, we iteratively add vertices as in Definition 3. G_2 and G_3 are defined symmetrically.

Lemma 12. *G_i is a planar triangulated graph and the Schnyder wood we obtain for G_i^σ is a minimal Schnyder wood for all $i \in \{1, 2, 3\}$.*

Proof (Sketch). We argued in Remark 1 and Lemma 4 that adding vertices in the manner of Definition 3 preserves the minimality of the Schnyder wood. Hence, in the following, we only consider the graph before we add those vertices. W.l.o.g. let $i = 1$. As described in Remark 1 and Lemma 9, the orientation and coloring of $\overline{\tau(G)}$ is a minimal Schnyder wood. First, observe that adding the 2-colored edges to $\overline{\tau(G)} - L_1$ does not create multi-edges by Lemma 2, i.e., for every pair of vertices there is at most one edge incident to both.

Second, we show that G_1 is indeed planar. Assume that G_1 is not planar. Then, there exists a facial cycle C in $\overline{\tau(G)} - L_1$ with four vertices $x, v, y, u \in C$ in that clockwise order such that there are 2-colored edges $xy, vu \in E(G_1)$. This implies that there are 2-colored paths P_{xy} and P_{vu} in $\overline{\tau(G)}$ in the interior of C connecting x with y and v with u, respectively. Those paths need to intersect. Let p be the last vertex of this intersection in the direction of color 2. At p Definition 1(iii) is violated, a contradiction. This implies that G_1 is planar.

Lemma 10 yields that our construction does not split 3-colored paths from a vertex to the root. Since we do only delete leaves of T_1, we also do not split 1-colored paths. And if we split a 2-colored path, then we patch this path with a 2-colored edge. These are the key observations in order to show that our construction yields a Schnyder wood. Finally, it is also possible to show that this Schnyder wood of G_1^σ is minimal.

In the following, we need to deal with multiple graphs and their Schnyder woods. If needed, we add a specifier. We refer for example by $r_1(G_1)$ to the root of the 1-colored tree of the Schnyder wood of G_1^σ.

Definition 5. *Define $H(G)$ to be the graph with a Schnyder wood obtained by the following procedure. Take G_1, G_2 and G_3. Identify the edges on the outer face, recolor and reorient those edges as follows. Identify $r_1(G_3)r_3(G_3)$ with $r_1(G_2)r_2(G_2)$, color it with color 1 and orient it towards $r_1(G_2) = r_1(G_3)$. Identify $r_2(G_3)r_3(G_3)$ with $r_1(G_1)r_2(G_1)$, color it with color 2 and orient it towards $r_2(G_1) = r_2(G_3)$. Identify $r_1(G_1)r_3(G_1)$ with $r_2(G_2)r_3(G_2)$, color it with color 3 and orient it towards $r_3(G_1) = r_3(G_2)$ (Figure 6). Delete redundant half-edges.*

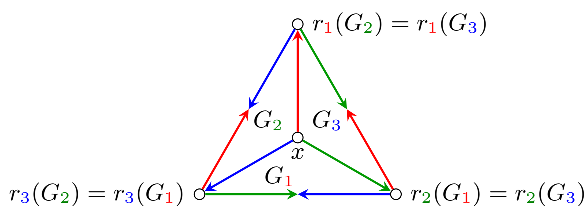


Figure 6: Illustration for Definition 5. Here $x = r_1(G_1) = r_2(G_2) = r_3(G_3)$.

Lemma 13. *$H(G)$ is triangulated and its orientation and coloring yields a minimal Schnyder wood of its suspension. Furthermore, depth$(G) \geq$ depth$(\overline{\tau(G)}) =$ depth$(H(G))$ and $|V(G)| \leq |V(H(G))|$.*

Proof. $H(G)$ is triangulated by construction. As observed in Lemma 12, the orientations and colorings of G_1, G_2 and G_3 are minimal Schnyder woods. Hence, by construction the orientation and coloring of $H(G)$ is a Schnyder wood. Assume, for the sake of contradiction, that there is a clockwise cycle C in $H(G)$. Observe that C cannot contain a vertex on the outer face of $H(G)$. Since all the outgoing edges of the vertex $x = r_1(G_1) = r_2(G_2) = r_3(G_3)$ end at vertices that are on the outer face of $H(G)$, C cannot contain x. Thus, C is completely contained in w.l.o.g. G_1, contradicting the minimality of the Schnyder wood of G_1^σ. And hence, the Schnyder wood of the suspension of $H(G)$ is minimal.

Let us consider the depth. By Lemma 9, $depth(G) \geq depth(\tau(G))$. By Definition 3, $depth(\tau(G)) = depth(\overline{\tau(G)})$. For $i \in \{2, 3\}$, by Definition 4, the depth of the i-colored tree of $\overline{\tau(G)}$ equals the depth of the i-colored tree in G_1 and, by Definition 5, the root of the i-colored tree of G_1 becomes the root of the i-colored tree of $H(G)$. Also, by Definition 4, the depth of the 1-colored tree of G_1 is by one smaller than the depth of the 1-colored tree of $\overline{\tau(G)}$. And, by Definition 5, the root of the 1-colored tree of G_1 has depth one in the 1-colored tree of $H(G)$. This holds symmetrically for G_2 and G_3. This yields that $depth(\overline{\tau(G)}) = depth(H(G))$, and altogether, $depth(G) \geq depth(H(G))$.

It remains to show that $|V(G)| \leq |V(H(G))|$. Using Lemma 11, we obtain that

$$\begin{aligned}
|V(H(G))| &\geq 1 + 3 + 3(|V(\overline{\tau(G)})| - 3) - \sum_{i=1}^{3} l_i^\circ(\overline{\tau(G)}) \\
&\geq 3|V(\overline{\tau(G)})| - 5 - 2|V(\overline{\tau(G)})| + 5 \\
&= |V(\overline{\tau(G)})|.
\end{aligned}$$

Here, $f^\circ(\overline{\tau(G)})$ and $l_i^\circ(\overline{\tau(G)})$ are the number of internal faces and the number of internal leaves of the i-colored tree of $\overline{\tau(G)}$, respectively. Obviously, we have $|V(\overline{\tau(G)})| \geq |V(G)|$, and thus, $|V(H(G))| \geq |V(G)|$.

Let A be a subgraph of G. Observe that we delete edges and vertices going from G to G_1, G_2 and G_3. Hence, there only remain subgraphs of A in G_1, G_2 and G_3. Let A' be the subgraph of $H(G)$ given by the union of the three subgraphs of A in G_1, G_2 and G_3. We say that A' *originates* from A. We define this relation to be transitive, i.e., if A originates from B and B originates from C, then A originates from C.

Let D be the subgraph of $\overline{\tau(G)}$ that is induced by its triangular outer face. Observe that the subgraph in $H(G)$ that originates from D is the complete graph K_4. We iterate this idea and finally obtain the following theorem.

Theorem 2. *Let G be a 3-connected planar graph with a minimal Schnyder wood. Then,*

$$depth(G) \geq \frac{1}{3\log_2(3)} \log_2(|V(G)|).$$

Proof. Define $H^s(G) := H(H^{s-1}(G))$ for $s \geq 2$ with $H^1(G) := H(G)$. The Schnyder wood we obtain for $H(G)$ is the minimal Schnyder wood, by Lemma 13. Thus,

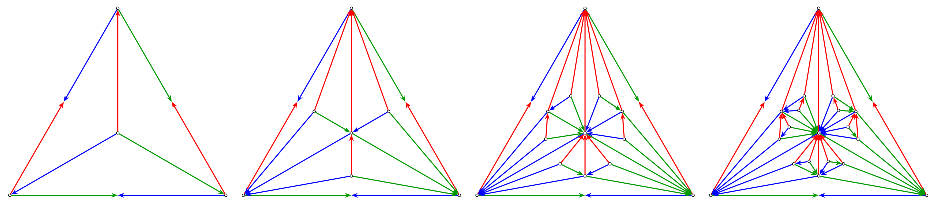


Figure 7: Let $depth(\overline{\tau(G)}) = 2$. From left to right we have the subgraphs of $H(G)$, $H^2(G)$, $H^3(G)$ and $H^4(G)$ that originate from the triangular outer face of $\overline{\tau(G)}$. Observe that the rightmost graph corresponds to G^2, compare Figure 2.

$H^s(G)$ is well-defined. Let $t = depth(\tau(G))$. Since $H(G)$ is already triangulated and all leaves of the three trees of the Schnyder wood have depth t, we have $\overline{\tau(H(G))} = H(G)$ and thus, by Lemma 13 and Definition 3, $depth(H^{3t-2}(G)) = depth(\tau(G)) = t$. One can show that $H^{3t-2}(G) = G^t$ (Figure 7). And hence, Lemma 6 and 13 yield

$$\begin{aligned} depth(G) &\geq depth(H(G)) \geq \ldots \geq depth(H^{3t-2}(G)) = depth(G^t) \\ &\geq \frac{1}{3\log_2(3)} \log_2(|V(G^t)|) = \frac{1}{3\log_2(3)} \log_2(|V(H^{3t-2}(G))|) \geq \ldots \\ &\geq \frac{1}{3\log_2(3)} \log_2(|V(G)|). \end{aligned}$$

Corollary 1. *Every 3-connected planar graph G on n vertices has an induced path of size at least $\lfloor 1/(3\log_2(3)) \log_2(n) \rfloor + 1$.*

Proof. Take a minimal Schnyder wood of G^σ. For a vertex $v \in V(G)$ and $i \in \{1, 2, 3\}$, the v-r_i-path $P_i(v)$ in the tree T_i is always induced. Assume, for the sake of contradiction, that there exists a vertex $v \in V(G)$ for which this does not hold. Then, there is a j-colored edge $e = xy$ in G with $x, y \in P_i(v)$, $i, j \in \{1, 2, 3\}$. By Lemma 1, T_i is a tree and hence $i \neq j$. Now, either $T_i^{-1} \cup T_j$ or $T_i \cup T_j$ has an oriented cycle, contradicting Lemma 2. Hence, Theorem 2 directly yields an induced path of size $\lfloor 1/(3\log_2(3)) \log_2(n) \rfloor + 1$.

5 Conclusion

In this paper, we gave a tight bound on the depth of a minimal Schnyder wood and used this bound to give a lower bound on the length of an induced path. Observe that the suspension of the graph G^k which we used to show that our lower bound is tight has a unique Schnyder wood for our choice of the outer face. Hence, if we want to exploit this method further, then we need to allow to choose the outer face.

References

1. Arocha, J.L., Valencia, P.: Long induced paths in 3-connected planar graphs. Discuss. Math. Graph Theory **20**(1), 105–107 (2000). https://doi.org/10.7151/dmgt.1110
2. Bonichon, N., Felsner, S., Mosbah, M.: Convex drawings of 3-connected plane graphs. Algorithmica **47**(4), 399–420 (2007)
3. Di Battista, G., Tamassia, R., Vismara, L.: Output-sensitive reporting of disjoint paths. Algorithmica **23**(4), 302–340 (1999). https://doi.org/10.1007/PL00009264
4. Di Giacomo, E., Liotta, G., Mchedlidze, T.: Lower and upper bounds for long induced paths in 3-connected planar graphs. Theoret. Comput. Sci. **636**, 47–55 (2016). https://doi.org/10.1016/j.tcs.2016.04.034
5. Erdős, P., Saks, M., Sós, V.T.: Maximum induced trees in graphs. J. Combin. Theory Ser. B **41**(1), 61–79 (1986). https://doi.org/10.1016/0097-3165(86)90115-9
6. Esperet, L., Lemoine, L., Maffray, F.: Long induced paths in graphs. European J. Combin. **62**, 1–14 (2017). https://doi.org/10.1016/j.ejc.2016.11.011
7. Felsner, S.: Geodesic embeddings and planar graphs. Order **20**, 135–150 (2003)
8. Felsner, S.: Convex drawings of planar graphs and the order dimension of 3-polytopes. Order **18**(1), 19–37 (2001). https://doi.org/10.1023/A:1010604726900
9. Felsner, S.: Geometric graphs and arrangements. Advanced Lectures in Mathematics, Friedr. Vieweg & Sohn, Wiesbaden (2004). https://doi.org/10.1007/978-3-322-80303-0
10. Felsner, S.: Lattice structures from planar graphs. Electron. J. Combin. **11**(1), 15–24 (2004)
11. Éric Fusy: Combinatorics of planar maps and algorithmic applications (Combinatoire des cartes planaires et applications algorithmiques). Ph.D. thesis, École Polytechnique, Palaiseau, France (2007), https://tel.archives-ouvertes.fr/pastel-00002931
12. Kant, G.: Drawing planar graphs using the lmc-ordering. In: Proceedings of the 33rd Annual Symposium on Foundations of Computer Science (FOCS'92). pp. 101–110 (1992). https://doi.org/10.1109/SFCS.1992.267814
13. Kant, G.: Drawing planar graphs using the canonical ordering. Algorithmica **16**(1), 4–32 (1996), http://dx.doi.org/10.1007/BF02086606
14. Ossona de Mendez, P.: Orientations bipolaires. Ph.D. thesis, École des Hautes Études en Sciences Sociales, Paris (1994)
15. Ortlieb, C.: Schnyder woods and long induced paths in 3-connected planar graphs. In: Proceedings of Latin American Theoretical Informatics Conference (2024), to appear
16. Ortlieb, C.: Toward Grünbaum's conjecture for 4-connected graphs (2024), under review
17. Ortlieb, C., Schmidt, J.M.: Toward Grünbaum's Conjecture. In: 19th Scandinavian Symposium and Workshops on Algorithm Theory (SWAT 2024). Leibniz International Proceedings in Informatics (LIPIcs), vol. 294, pp. 37:1–37:17. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl, Germany (2024). https://doi.org/10.4230/LIPIcs.SWAT.2024.37
18. Schnyder, W.: Embedding planar graphs on the grid. In: Proceedings of the first annual ACM-SIAM symposium on Discrete algorithms. pp. 138–148 (1990)