Contractible Sets in 3-Connected Planar Graphs via Schnyder Woods

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Abstract

A connected vertex subset X of a graph G is *contractible* if $G - X$ is 2-connected. A far-reaching conjecture by McCuaig and Ota states that for every $k \geq 2$ there is a number $f(k)$ such that every 3-connected graph of order at least $f(k)$ contains a contractible set of size k. Here, we study the number of contractible sets of different sizes in 3-connected graphs that are planar and utilize Schnyder woods for this task.

It is well-known that Schnyder woods can be used to find three compatible ordered path partitions. We use the latter to prove that every 3-connected planar graph G contains $|E(G)|/3 - 1$ contractible sets of different size. This bound is exponentially better than the best-known ones before and in addition tight. Our results can be seen as support to an affirmative solution of the McCuaig-Ota conjecture for planar graphs.

1 Introduction

We are interested in gaining more insight into the structure of 3-connected planar graphs. One traditional way to do this is to identify the edges whose contractions preserve the graph to be 3-connected. Somewhat more general, let a vertex subset X of a graph G be *contractible* if $G[X]$ is connected and the graph G/X, i.e. the graph obtained by contracting the edges of $G[X]$, is 3-connected. Equivalently, X is contractible if and only if $G[X]$ is connected and $G - X$ is 2-connected [18].

In 1994, McCuaig and Ota [22] proposed their far-reaching conjecture that for *every* $k \geq 2$ there is a number $f(k)$ such that every 3-connected graph of order at least $f(k)$ contains a contractible set of size k. This conjecture would, if proven true, allow for a new reduction method of 3-connected graphs. Such reduction methods are one of the main concerns of graph connectivity theory; they are in particular used in inductive proofs or the classification of graph classes. Hence, every step towards this conjecture sheds more light on the structure of 3-connected graphs. We take such a step by proving a lower bound on the number of contractible sets of different sizes in planar graphs; our methohds use Schnyder woods and ordered path partitions.

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Contractible Sets The study of contractible sets of 3-connected graphs started in 1961. Tutte [25] proved that every 3-connected graph except for K_4 contains a *contractible edge* (i.e. an edge whose contraction leaves a 3-connected graph) and hence a contractible set of size $k = 2$. McCuaig and Ota [22] generalized this by showing that every 3-connected graph of order at least 9 contains a contractible set of size $k = 3$, and proposed the above mentioned conjecture. These two results [25, 22] show that $f(2) = 5$ and $f(3) = 9$, respectively; these values are sharp. Kriesell [16] proved that the conjecture holds for several specific graph classes (planar triangulations, a superclass of Halin graphs and line graphs of 6-edge-connected graphs) and in general for $k = 4$ by showing $f(4) = 8$, which is sharp. He also showed that with some additional assumptions the conjecture holds for the case $k = 5$ [18] and gave a lower bound on the number of contractible sets of size $k = 3$ [19]. Recently, in 2022, Vlasova [26] could show that conjecture holds for $k = 5$ without any additional assumptions. In the same year, Karol [13] gave proof that the conjecture holds for graphs having minimum degree at least $(2k + 7)/3$ (he also claims a proof for $k = 6$).

If we allow that contractible sets are not necessarily connected, the McCuaig-Ota-Conjecture is true, as shown by Kriesell [17]. A generalization of this has been investigated by Mader [21] to graphs of higher connectivity: for every $l \geq 4$, $k \geq 1$ there exists a number $f_l(k)$ such that every l-connected graph G of order at least $f_l(k)$ contains a vertex set X of size k such that $G - X$ is $(l - 2)$-connected. Observe that X does not need to be connected in this statement.

While the original conjecture of McCuaig and Ota is still wide open, weaker statements have been studied. The conjecture implies that any 3-connected graph of order at least $f(k)$ contains a contractible set of size l for every $l \leq k$. This motivates the slightly less elusive question, whether one may prove density results about the numbers l, for which a contractible set of size l exists. Recently, Karpov [14] proved that every 3-connected graph of order at least $2l + 1$ contains a contractible set whose size is in the interval $[l, 2l - 4]$. For $l = 5$, for example, this implies that every 3-connected graph of order at least 11 contains a contractible set of size 5 or 6. Independently, Kriesell showed 2018 such a result for the interval $[l, 2l]$ [20].

We measure progress to a solution of the McCuaig-Ota conjecture by counting how many contractible sets of different sizes can be proven. Karpov's result provides the following lower bound on the number of (pairwise) differently sized contractible sets as follows. Observe that for all $x \in \mathbb{N}$ the intervals $I_i := [2^{i-1} + 3, 2^i + 2]$ with $i \in \{1, \ldots, x\}$ partition the interval $[4, 2^x + 2]$. We also have $I_i = [k, 2k - 4]$ for $k = 2^{i-1} + 3$. Hence, Karpov's result implies that any 3-connected graph of order $n \geq 2^x + 7$ contains at least x, and thus $\log_2(n - 7)$, differently sized contractible sets (for $n \geq 9$). This is the best bound known so far. On a side note, we want to remark that when we ask for the number of *contractible subgraphs* instead of contractible sets, much better lower bounds are known [23, Application 4]; the reason for this is that two different contractible subgraphs may differ only on their edge set while two different contractible sets need to differ on the set of vertices.

Like Kriesell [16] and Karol [13], we consider the conjecture of McCuaig and Ota for a restricted class of graphs, in our case for 3-connected planar graphs G. We prove the lower bound $|E(G)|/3 - 1$ on the number of contractible sets of different size. This bound is exponentially better than the one we deduced from Karpov's result and supports evidence that the McCuaig-Ota conjecture might hold for planar graphs. Moreover, this bound is tight; we give a proof of the bound and its tightness in Section 3.

Since 3-connected graphs G satisfy $|E(G)| \geq \frac{3}{2}|V(G)|$, this bound implies at least $|V(G)|/2 - 1$ differently sized contractible sets. We use Schnyder woods and so called compatible ordered path partitions for our result and show that these imply, under certain conditions, contractible sets of different size.

Schnyder Woods and Ordered Path Partitions It is well-known that 3-connected planar graphs admit Schnyder woods, which have many applications and implications for planarity and graph drawing, see for example [24, 1, 2, 5, 8, 9, 10, 15, 28, 27, 4]. A Schnyder wood of a 3-connected planar graph is a triple (T_1, T_2, T_3) of oriented spanning trees such that every edge is contained in at least one and at most two trees and certain conditions on the edge order around each vertex hold (we give a precise definition in Section 2). For every pair (T_i, T_j), $i \neq j$, the edges that are contained in both T_i and T_j induce paths. The third tree T_l, $l \notin \{i, j\}$ may then be used to arrange these paths in an order. This then yields a so-called *ordered path partition compatible* to the Schnyder wood (T_1, T_2, T_3), which is nothing else than an ordered partition of the vertices of a graph into vertex sets of paths. By choosing (T_i, T_j) in all possible ways, every Schnyder wood generates three such compatible ordered path partitions; Figure 1 provides an illustration. Ordered path partitions may also be seen as generalizations of canonical orderings [6, 7, 2]. We give a formal definition of Schnyder woods, ordered path partitions and show how these two concepts relate in Section 2.

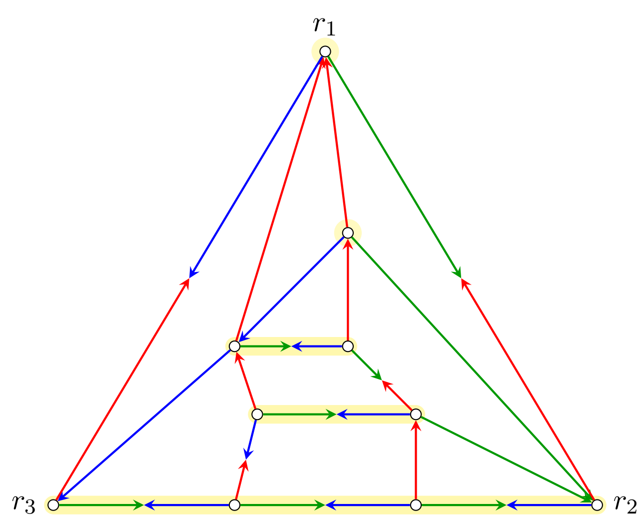


Figure 1: A 3-connected planar graph and one of its Schnyder woods. The three trees of the Schnyder wood are depicted in colors red, blue and green, and are oriented towards the roots r_1, r_2 and r_3, respectively. One of the three compatible ordered path partitions, namely the one partitioning into green-blue paths, is marked in fat yellow.

2 Schnyder Woods and Ordered Path Partitions

We define Schnyder woods and ordered path partitions, show how these two relate and give some basic properties that we need for the proofs. We only consider simple undirected graphs. A graph is *plane* if it is planar and embedded into the Euclidean plane.

2.1 Schnyder Woods

Let r_1, r_2 and r_3 be three vertices of the outer face boundary of a plane graph G in clockwise order and $\sigma = \{r_1, r_2, r_3\}$. The *suspension* G^σ of G is the graph obtained from G by adding at each vertex of $\{r_1, r_2, r_3\}$ a half-edge, an arc pointing into the outer face. We call r_1, r_2 and r_3 *roots*.

Definition 1. Let $\sigma = \{r_1, r_2, r_3\}$ and G^σ be the suspension of a 3-connected plane graph G. A *Schnyder wood* of G^σ is an orientation and coloring of the edges of G^σ (including the half-edges) with the colors 1, 2, 3 (red, green, blue) such that

(a) Every edge e is oriented in one direction (we say e is *unidirected*) or in two opposite directions (we say e is *bidirected*). Every direction of an edge is colored with one of the three colors 1, 2, 3 (we say an edge is *i-colored* if one of its directions has color i) such that the two colors i and j of every bidirected edge are distinct (we call such an edge *i-j-colored*). Throughout the paper, we assume modular arithmetic on the colors 1, 2, 3 such that $i+1$ and $i-1$ for a color i are defined as $(i \mod 3)+1$ and $(i+1 \mod 3)+1$ respectively. For a vertex v, an incident uni- or bidirected edge is *ingoing* (i-colored) in v if it has a direction (of color i) that is directed toward v, and *outgoing* (i-colored) of v if it has a direction (of color i) that is directed away from v.

(b) For every color i, the half-edge at r_i is unidirected, outgoing and i-colored.

(c) Every vertex v has exactly one outgoing edge of every color. The outgoing 1-, 2-, 3-colored edges e_1, e_2, e_3 of v occur in clockwise order around v. For every color i, every ingoing i-colored edge of v is contained in the clockwise sector around v from e_{i+1} to e_{i-1} (see Figure 2).

(d) No inner face boundary contains a directed cycle (disregarding possible opposite directions) in one color.

For an illustration of Definition 1, see Figure 3. For ease of notation, let $N^l(v)$ for a vertex v be the neighbors of v that share an edge having two colors with v. Also, if the choice of the roots is clear from context, we might speak of Schnyder woods of a graph G instead of Schnyder woods of the suspension of G. The below lemmas give some basic properties of Schnyder woods.

Lemma 2 (Felsner [10])**.** *Every 3-connected plane graph has a Schnyder wood.*

For a Schnyder wood and color i, let T_i be the directed graph that is induced by the directed edges of color i. The following result justifies the name of Schnyder woods.

Lemma 3 (Schnyder[24], Felsner[11])**.** *For every color i of a Schnyder wood of a graph G, T_i is a directed spanning tree of G in which all edges are oriented towards the root r_i.*

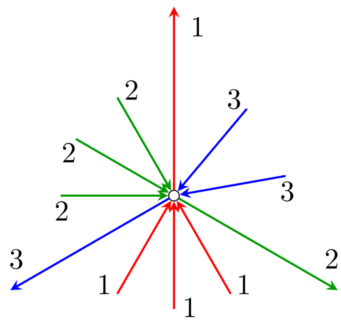


Figure 2: Properties of Schnyder woods. Condition 1(c) at a vertex. Color 1 is depicted in red, color 2 in green and color 3 in blue as for the rest of the paper.

2.2 Ordered Path Partitions

Schnyder woods are closely related to the concept of ordered path partitions. In the following we give the definition of ordered path partitions and show how the two concepts relate.

Definition 4. Let G be a plane graph with three vertices r_1, r_2 and r_3 in that clockwise order on the outer face. An *ordered path partition* $\mathcal{P} = (P_0, \ldots, P_s)$ of G with *base-pair* (r_j, r_{j+1}), $j \in \{1, 2, 3\}$, is an ordered partition of $V(G)$ into the vertex sets of induced paths such that the following holds for every $i \in \{1, \ldots, s-1\}$, where $V_k := \bigcup_{q=0}^{k} V(P_q)$.

(a) P_0 consists of the vertices of the counterclockwise path from r_{j+1} to r_j on the outer face boundary, and $P_s = \{r_{j+2}\}$.

(b) Each vertex in P_i has a neighbor in $V(G) \setminus V_i$.

(c) $G[V_i] \cup \{r_{j+1}r_j\}$ is 2-connected.

(d) Each vertex in V_i has at most one neighbor in P_{i+1}.

An ordered path partition $\mathcal{P} = (P_0, \ldots, P_s)$ with base pair (r_j, r_{j+1}) defines a Schnyder wood by the following procedure [3, 12]: For each vertex $x \in P_l \setminus \{r_1, r_2, r_3\}$ and $l \in \{0, \ldots, s\}$, the three outgoing edges of x end (1) at its counterclockwise neighbor on the outer face of $G[V_l]$ ($(j+1)$-colored edge), (2) at its clockwise neighbor on the outer face of $G[V_l]$ (j-colored edge) and (3) at its neighbor which is on the path of maximal index ($(j+2)$-colored edge). The root vertices r_1, r_2, r_3 are handled by the same procedure, but omitting the outgoing j-colored edge for r_j and instead adding an outgoing j-colored half-edge for every $j \in \{1, 2, 3\}$. An ordered path partition and a Schnyder wood that are in this relation are called *compatible* with each other.

The conversion can also be done backwards. The vertex sets of the inclusion-wise maximal j-$(j+1)$-colored paths of any Schnyder wood S of a suspension G^σ, $j \in \{1, 2, 3\}$ partition the vertex set of G [2, 1]. As shown in [2, Theorem 5], [1, Lemma 1 (WADS version), Section 2.2 (arXiv Version)], one can order those paths such that the resulting ordered path partition is compatible with S (and has the base pair (r_j, r_{j+1})). We denote a compatible ordered path partition with base pair (r_j, r_{j+1}) by $\mathcal{P}^{j,j+1}$.

By 4(a) and 4(b), G contains for every i and every vertex $v \in P_i$ a path from v to r_{j+2} that intersects V_i only in v. Since G is plane, this implies that every path P_i is embedded into the outer face of $G[V_{i-1}]$ for every $1 \leq i \leq s$. A path which consists of a single vertex is called a *singleton*.

3 Contractible Sets

In this section, we give a tight lower bound on the number of contractible sets of different sizes in a 3-connected planar graph in terms of the number of edges. We use structural properties of Schnyder woods to attain this bound. In the following lemma, we show how to obtain contractible sets from an ordered path partition. This lemma is essential for the proof of our main result.

Lemma 5. *Every 3-connected planar graph G has a Schnyder wood S such that for every ordered path partition $\mathcal{P}$ compatible with S, G contains at least $|\mathcal{P}| - 2$ contractible sets of different size.*

Proof. Assume that $|V(G)| > 3$ as otherwise the claim is trivial. First, we show that G has a triangular face or G has a vertex of degree 3. So assume that G does not have a triangular face. We now show that, under this assumption, G has a vertex of degree 3. Fix an embedding of G. Let f be the number of faces of G. Every edge is incident to two faces and hence $2|E(G)| = \sum_{F \text{ face of } G} |F| \geq 4f$. Plugging this into Euler's polyhedral formula yields that $|E(G)| \leq 2|V(G)| - 4$. Combining this with the Handshake lemma, we obtain $\sum_{v \in V(G)} \deg(v) = 2|E(G)| \leq 4|V(G)| - 8$. This implies that G contains a vertex of degree 3. We may therefore distinguish the following two cases.

Case 1: G has a triangular face.

Choose an embedding of G such that the outer face is triangular. Choose a Schnyder wood S of G, and let $\mathcal{P} = (P_0, \ldots, P_s)$ be any ordered path partition that is compatible with S. W.l.o.g. let r_1, r_2 and r_3 be the roots of S such that (r_2, r_3) is the base-pair of $\mathcal{P}$. Remember that $V_k := \bigcup_{q=0}^{k} V(P_q)$ for $k \in \{0, \ldots, s\}$.

Since the outer face of G is triangular, we obtain that $r_2 r_3 \in E(G)$. Hence, for every $0 < i < s$, the graph $G[V_i]$ is 2-connected, by Definition 4(c). Also, by Definition 4(a) and (b), $G[V(G) \setminus V_i]$ is connected for every $0 < i < s$. This implies that G contains $s - 1 = |\mathcal{P}| - 2$ contractible sets of different size.

Case 2: G contains a vertex v of degree three.

Choose an embedding of G such that v is on the outer face. Choose the Schnyder wood S of G with roots $r_1 := v$, r_2 and r_3 such that r_2 and r_3 are the clockwise and counterclockwise neighbor of r_1 on the outer face, respectively. Let $\mathcal{P} = (P_0, \ldots, P_s)$ be any ordered path partition of G that is compatible with S.

For the compatible ordered path partitions $\mathcal{P}^{1,2}$ and $\mathcal{P}^{3,1}$, we may proceed as in Case 1. In the remaining case, $\mathcal{P} = \mathcal{P}^{2,3}$. The first element P_0 of $\mathcal{P}$ is the counterclockwise path from r_3 to r_2 on the outer face boundary. We delete the last element $P_s = \{r_1\}$

of $\mathcal{P}$ and prepend the new element $P_{-1} := \{r_1\}$ to $\mathcal{P}$. Then $P_{-1} \cup P_0$ induces a cycle in G (see Figure 3). Since r_1 has degree 3, only one vertex of $P_1 \cup \ldots \cup P_{s-1}$ can have a neighbor in P_s. This and Definition 4(b) yield that P_{s-1} is a singleton and also the only path that has a neighbor in P_s. Hence, by Definition 4(b), $P_i \cup \ldots \cup P_{s-1}$ induce a connected graph for every $i \in \{1, \ldots, s-1\}$. By Definition 4(c), for every $1 \le i \le s-1$, $P_{-1} \cup \cdots \cup P_{i-1}$ induces a 2-connected graph. This implies that G contains $|\mathcal{P}| - 2$ contractible sets of different size.

In both cases, we showed that we obtain $|\mathcal{P}| - 2$ contractible sets of different sizes from an ordered path partition $\mathcal{P}$ of G. Furthermore, this family of contractible sets is such that a member of the family contains all the members that are smaller than itself. □

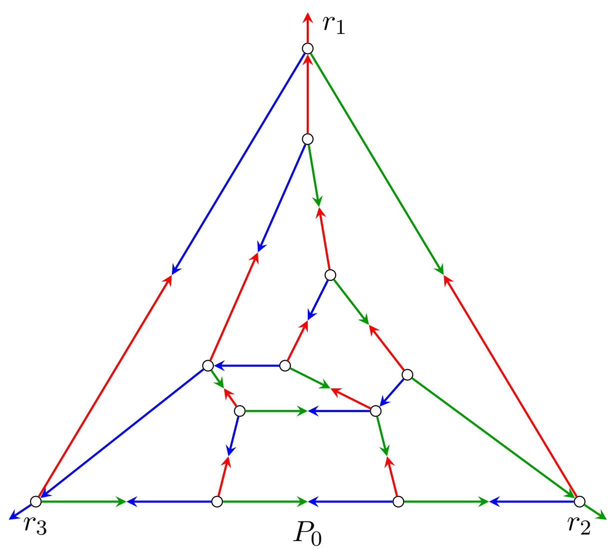


Figure 3: A Schnyder wood of the suspension of a 3-connected planar graph having a vertex of degree three on the outer face boundary.

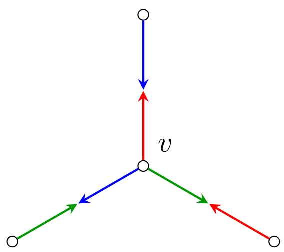
Theorem 6. *Let G be a 3-connected planar graph. Then G has at least $\frac{|E(G)|}{3} - 1$ contractible sets of different sizes.*

Proof. Choose an embedding and a Schnyder wood S of G according to Lemma 5. So G has at least $|\mathcal{P}| - 2$ contractible sets of different sizes for every compatible ordered path partition $\mathcal{P}$. Let r_1, r_2 and r_3 be the roots of S and $i \in \{1, 2, 3\}$. Remember that the maximal i-$(i+1)$-colored paths form an ordered path partition $\mathcal{P}^{i,i+1}$.

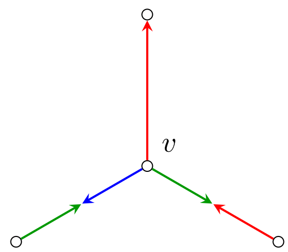
For our counting argument, we need the following definition. For every vertex $v \in V(G)$, we define

$$v_{i,i+1} := \begin{cases} 1/2 & \text{if } v \text{ is the end of a non-trivial maximal } i\text{-}(i+1)\text{-colored path,} \\ 1 & \text{if } v \text{ is a singleton in } \mathcal{P}^{i,i+1}, \\ 0 & \text{otherwise.} \end{cases}$$

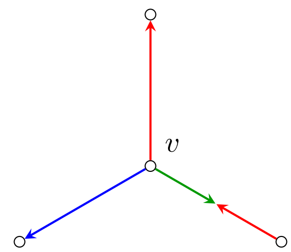
Naturally, we obtain that $\sum_{v \in V(G)} v_{i,i+1} = |\mathcal{P}^{i,i+1}|$, for every $i \in \{1, 2, 3\}$. Observe that, for every vertex $v \in V(G) \setminus \{r_1, r_2, r_3\}$, we have $\{v_{1,2}, v_{3,1}, v_{2,3}\} \in \{\{1, 1, 1\}, \{1, 1, 1/2\}, \{1, 1/2, 1/2\}, \{1/2, 1/2, 1/2\}, \{1, 1/2, 0\}, \{1, 1, 0\}\}$, see Figure 4. In fact, this is a lengthy case distinction. In Figure 44i, the three outgoing edges of v are all bidirected such that no two have the same two colors. In Figure 44ii, one of the three outgoing edges at v is unidirected and the remaining two do not have the same two colors. In Figure 44iii, exactly two of the three outgoing edges are unidirected. In Figure 44iv, the three outgoing edges of v are all unidirected. In Figure 44v, one outgoing edge of v is unidirected and the other two have the same two colors. In Figure 44vi, all the outgoing edges of v are bidirected and two of those have the same colors. By Definition 1(c), the three outgoing edges at v cannot all have the same two colors. So this case distinction is exhaustive.



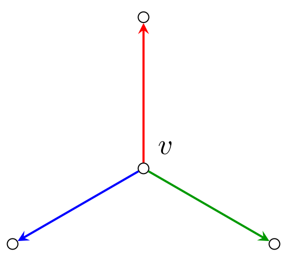
(i) 3 ends, $(v_{1,2}, v_{3,1}, v_{2,3}) = (1/2, 1/2, 1/2)$



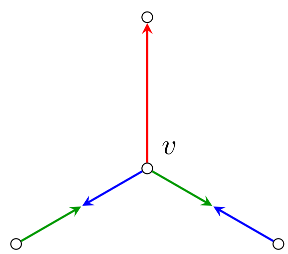
(ii) 2 ends, 1 singleton, $(v_{1,2}, v_{3,1}, v_{2,3}) = (1/2, 1, 1/2)$



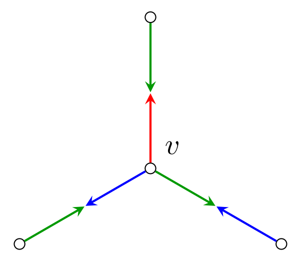
(iii) 1 end, 2 singletons, $(v_{1,2}, v_{3,1}, v_{2,3}) = (1/2, 1, 1)$



(iv) 3 singletons, $(v_{1,2}, v_{3,1}, v_{2,3}) = (1, 1, 1)$



(v) 2 singletons, $(v_{1,2}, v_{3,1}, v_{2,3}) = (1, 1, 0)$



(vi) 1 end, 1 singleton, $(v_{1,2}, v_{3,1}, v_{2,3}) = (1/2, 1, 0)$

Figure 4: The six possible cases for $\{v_{1,2}, v_{3,1}, v_{2,3}\}$ (up to symmetry).

Hence, for every vertex $v \in V(G) \setminus \{r_1, r_2, r_3\}$, we have $\sum_{i=1}^{3} v_{i,i+1} = 3/2 + o_v/2$, with o_v being the number of outgoing unidirected edges at v. Using Figure 4, this is easy to verify. So consider the root vertices. We know that r_1 is the end of a maximal 1-2-colored path, the end of a maximal 3-1-colored path and a singleton in $\mathcal{P}^{2,3}$. This holds symmetrically for r_2 and r_3.

Every vertex $v \in V(G) \setminus \{r_1, r_2, r_3\}$ has $\deg(v) - 3$ unidirected ingoing edges. A vertex $v \in \{r_1, r_2, r_3\}$ has $\deg(v) - 2$ such edges. And every vertex which is unidirected outgoing at a vertex of $V(G) \setminus \{r_1, r_2, r_3\}$ needs to be unidirected ingoing at a vertex of $V(G)$. Hence

$$\begin{aligned}
\sum_{i=1}^{3} |\mathcal{P}^{i,i+1}| &= \sum_{i=1}^{3} \sum_{v \in V(G)} v_{i,i+1} = \sum_{v \in V(G)} \sum_{i=1}^{3} v_{i,i+1} \\
&= 6 + \sum_{v \in V(G) \setminus \{r_1, r_2, r_3\}} \left(\frac{3}{2} + \frac{o_v}{2} \right) \\
&= \frac{3|V(G)|}{2} + \frac{3}{2} + \sum_{v \in V(G) \setminus \{r_1, r_2, r_3\}} \frac{o_v}{2} \\
&= \frac{3|V(G)|}{2} + \frac{3}{2} + \frac{3}{2} + \sum_{v \in V(G)} \frac{\deg(v) - 3}{2} \\
&= 3 + \frac{1}{2} \sum_{v \in V(G)} \deg(v) \\
&= 3 + |E(G)|.
\end{aligned}$$

By pigeonhole principle, there exists $\mathcal{P}^{i,i+1}$ with $|\mathcal{P}^{i,i+1}| \geq \left\lceil 1 + \frac{|E(G)|}{3} \right\rceil$. By Lemma 5, this gives rise to at least $\left\lceil 1 + \frac{|E(G)|}{3} \right\rceil - 2 \geq \frac{|E(G)|}{3} - 1$ contractible sets of different size. $\square$

Remark. Here, we explicitly calculate the number of contractible sets of different sizes for two graphs and compare this to our lower bound of Theorem 6.

Let H be a triangulated graph of order n. Then $|E(H)| = 3n - 6$. And, by Theorem 6, there are at least $|E(H)|/3 - 1 = n - 3$ contractible sets of different size. For a contractible set X we have by definition that $H - X$ is 2-connected. So there are at least three vertices in $H - X$ and we obtain that there are at most $n - 3$ contractible sets of different sizes in H. Thus H has exactly $n - 3 = |E(H)|/3 - 1$ such sets. This yields that the lower bound of Theorem 6 is tight. Also, this shows that McCuaig's and Ota's conjecture is true for triangulated planar graphs, a fact first proven by Kriesell [16].

Let us consider a graph that does not have a triangular face. For even l, let $G = C_l \times K_2$ be the circular ladder graph. G is 3-connected, planar and has $3l/2 - 3$ contractible sets of different size. We show how to obtain this number. For every $k \in \{1, \ldots, l-2\}$, paths with k vertices that have only vertices on one of the cycles C_l yield contractible sets of size k. A cycle C_l itself is a contractible set of size l. For all $i \in \{1, \ldots, l-2\}$, the subgraphs $P_i \times K_2$ yield contractible sets of size a with a even and $2 \leq a \leq 2l - 4$. This sums up to $3l/2 - 3$ contractible sets of different size. G has $2l$ vertices and $3l$ edges. So we have $|E(G)|/2 - 3 = 3|V(G)|/4 - 3$ contractible sets of different size. Theorem 6 yields a lower bound of $|E(G)|/3 - 1 = |V(G)|/2 - 1$ and is therefore off by a gap of $|E(G)|/6 - 2 = |V(G)|/4 - 2$.

In the following, we want to briefly show that if we want a more exact bound for the number of contractible sets of different sizes of G, then counting the paths of an ordered path partition does not suffice. We show that, every ordered path partition of G has at most $n/2 + 2$ paths. Remember that $V_k := \bigcup_{q=0}^{k} V(P_k)$. Observe that, for an ordered path partition $\mathcal{P} = (P_0, \ldots, P_s)$ of G, $G[V_{i+1}]$ has at least one face more than $G[V_i]$ for all $i \in \{0, \ldots, s-1\}$. G itself hast $n/2 + 2$ faces and thus $\mathcal{P}$ has at most $n/2 + 2$ paths.

Hence, for a better bound we do not need to refine the counting argument of Theorem 6, but need to change the method completely.

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