Theoretical Computer Science 1056 (2025) 115551

Contents lists available at ScienceDirect

Theoretical Computer Science

journal homepage: www.elsevier.com/locate/tcs

Toward Grünbaum's conjecture bounding vertices of degree 4

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ARTICLE INFO

Handling Editor: Dr. Pinyan Lu

Keywords:
Planar graph
Spanning tree
Maximum degree
Schnyder wood

ABSTRACT

Given a spanning tree T of a planar graph G, the *co-tree* of T is the spanning tree of the dual graph G^* with edge set $(E(G) - E(T))^*$. Grünbaum conjectured in 1970 that every planar 3-connected graph G contains a spanning tree T such that both T and its co-tree have maximum degree at most 3.

While Grünbaum's conjecture remains open, Schmidt and the author recently improved the upper bound on the maximum degree from 5 (Biedl 2014) to 4.

In this paper, we modify this approach taking a further step towards Grünbaum's conjecture. We again obtain a spanning tree T such that both T and its co-tree have maximum degree at most 4 and, additionally, an upper bound on the number of vertices of degree 4 of T and its co-tree.

1. Introduction

Let a k-*tree* be a spanning tree whose maximum degree is at most k. In 1966, Barnette proved the fundamental theorem that every planar 3-connected graph contains a 3-tree [1]. Both assumptions of Barnette's theorem are essential in the sense that the statement fails for arbitrary non-planar graphs (as the arbitrarily high degree in any spanning tree of the complete bipartite graphs $K_{3,n-3}$ shows) as well as for graphs that are not 3-connected (as the planar graphs $K_{2,n-2}$ show).

Since then, Barnette's theorem has been extended and generalized in several directions [2–9]. Perhaps one of the most severe strengthenings is a long-standing and to the best of our knowledge still open conjecture made by Grünbaum in 1970. Since the planar dual G^* of every (simple) planar 3-connected graph G is again planar and 3-connected, G^* contains a 3-tree as well. By the well-known cut-cycle duality, any spanning tree T of G implies that also $(V^*, (E(G) - E(T))^*)$ is a spanning tree $\neg T^*$ of G^*; we call $\neg T^*$ the *co-tree* of T. Taking the best of these two worlds, Grünbaum made the following conjecture.

Conjecture (Grünbaum [10, p. 1148], 1970). *Every planar 3-connected graph G contains a 3-tree T whose co-tree $\neg T^*$ is also a 3-tree.*

While Grünbaum's conjecture is to the best of our knowledge still unsolved, progress has been made by Biedl [7], who proved the existence of a 5-tree whose co-tree is a 5-tree. Exploiting insights into the structure of Schnyder woods, Schmidt and the author [11] proved the existence of a 4-tree whose co-tree is a 4-tree and for 4-connected graphs, the author [12] showed that there exists a 3-tree whose co-tree is a 4-tree. In this paper, we improve on the result for 3-connected graphs by additionally giving upper bounds on the number of vertices of degree 4 in both the tree and the co-tree. The approach is similar to the one in [11]. We use a different candidate graph, show that it meets all necessary conditions and then apply a method similar to the one in [11]. We observe that only under specific local conditions vertices of degree 4 might arise. Thus, we are able to count them.

We fix a minimal Schnyder wood S of G. S gives rise to a Schnyder wood of the suspended dual (a graph that differs from the dual graph only on the outer face). Also, every Schnyder wood has three compatible ordered path partitions, denoted by $\mathcal{P}^{j,j+1}$,

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https://doi.org/10.1016/j.tcs.2025.115551
Received 22 November 2024; Received in revised form 1 August 2025; Accepted 8 September 2025
Available online 11 September 2025

$j \in \{1,2,3\}$. We define and explain those concepts in Section 2. The upper bound on the number of degree-4-vertices in T and $\neg T^*$ is then given by $\min\{B'_{2,3}, n'+2-A'_{3,1}, n'+2-A'_{1,2}\}-1$ and $\min\{B_{2,3}, n-A_{3,1}, n-A_{1,2}\}-1$, respectively. Here n (n') is the order of the primal (dual) graph, $B_{j,j+1}$ ($B'_{j,j+1}$) is the number of singletons in $\mathcal{P}^{j,j+1}$ of the primal (dual) graph and $A_{j,j+1}$ ($A'_{j,j+1}$) is the number of paths in $\mathcal{P}^{j,j+1}$ of the primal (dual) graph.

Our arguments work symmetrically for any choice of two colors. Thus, we obtain for example that if one of the compatible ordered path partitions of the minimal Schnyder wood of the dual graph has only one singleton (which is the smallest possible number of singletons), then the primal graph has a 3-tree such that its co-tree has maximum degree at most 4.

We discuss Schnyder woods, their lattice structure and ordered path partitions in Section 2 and our main result in Section 3.

2. Schnyder woods and ordered path partitions

We only consider simple undirected graphs. A graph is *plane* if it is planar and embedded into the Euclidean plane without intersecting edges. The *neighborhood of a vertex set* A is the union of the neighborhoods of vertices in A. Although parts of this paper use orientation on edges, we will always let vw denote the undirected edge $\{v,w\}$.

2.1. Schnyder woods

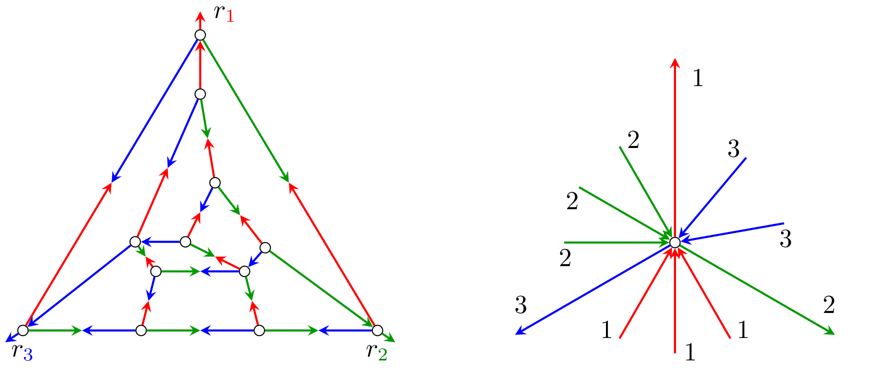
Let $\sigma := \{r_1, r_2, r_3\}$ be a set of three vertices of the outer face boundary of a plane graph G in clockwise order (but not necessarily consecutive). We call r_1, r_2 and r_3 *roots*. The *suspension* G^σ of G is the graph obtained from G by adding at each root of σ a half-edge pointing into the outer face. With a little abuse of notation, we define a *half-edge* as an arc starting at a vertex but with no defined end vertex. A plane graph G is *σ-internally 3-connected* if the graph obtained from the suspension G^σ of G by making the three half-edges incident to a common new vertex inside the outer face is 3-connected. Note that the class of σ-internally 3-connected plane graphs properly contains all 3-connected plane graphs.

Definition 1 (Felsner [13]). Let $\sigma = \{r_1, r_2, r_3\}$ and G^σ be the suspension of a σ-internally 3-connected plane graph G. A *Schnyder wood* of G^σ is an orientation and coloring of the edges of G^σ (including the half-edges) with the colors 1,2,3 (red, green, blue) such that

(i) Every edge e is oriented in one direction (we say e is *unidirected*) or in two opposite directions (we say e is *bidirected*). Every direction of an edge is colored with one of the three colors 1,2,3 (we say an edge is *i-colored* if one of its directions has color i) such that the two colors i and j of every bidirected edge are distinct (we call such an edge *$i-j$-colored*). Throughout the paper, we assume modular arithmetic on the colors 1,2,3.
(ii) For every color i, the half-edge at r_i is unidirected, outgoing and i-colored.
(iii) Every vertex v has exactly one outgoing edge of every color. The outgoing 1-, 2-, 3-colored edges e_1, e_2, e_3 of v occur in clockwise order around v. For every color i, every incoming i-colored edge of v is contained in the clockwise sector around v from e_{i+1} to e_{i-1} (see Fig. 1(ii)).
(iv) No inner face boundary contains a directed cycle in one color.

See Fig. 1(i) for illustration.

For a Schnyder wood and color i, let T_i be the directed graph that is induced by the directed edges of color i. The following result justifies the name of Schnyder woods.



(i) A Schnyder wood of the suspension of a 3-connected plane graph.

(ii) Example for Definition 1(iii) at a vertex in a Schnyder wood. The incoming edges in color i are in the clockwise sector between the outgoing edge in color $i+1$ and the outgoing edge in color $i-1$.

Fig. 1. Illustrations for the definition of Schnyder woods.

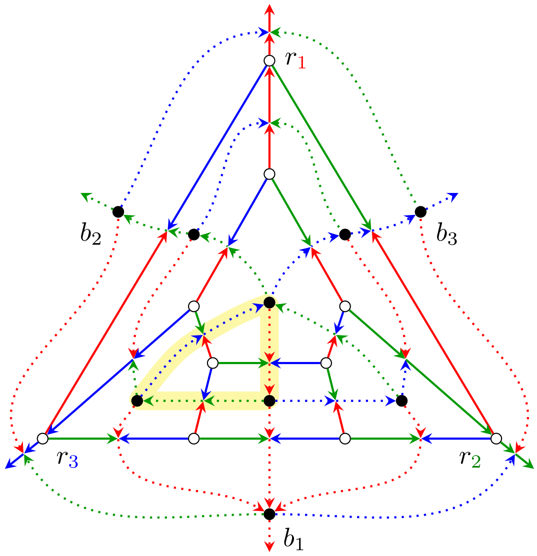


Fig. 2. The completion of G obtained by superimposing G^σ and its suspended dual G^{σ^*} (the latter depicted with dotted edges). The primal Schnyder wood is not the minimal element of the lattice of Schnyder woods of G, as this completion contains a clockwise directed cycle (marked in yellow). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Lemma 1 ([14,15]). *For every color i of a Schnyder wood of G^σ, T_i is a directed spanning tree of G in which all edges are oriented towards the root r_i.*

For a directed graph H, we denote by H^{-1} the graph obtained from H by reversing the direction of all its edges.

Lemma 2 (Felsner [16]). *For every $i \in \{1, \ldots, 3\}$, $T_i^{-1} \cup T_{i+1}^{-1} \cup T_{i+2}$ is acyclic.*

2.2. Dual Schnyder woods

Let G be a σ-internally 3-connected plane graph. Any Schnyder wood of G^σ induces a Schnyder wood of a slightly modified planar dual of G^σ in the following way [17,18] (see [19, p. 30] for an earlier variant of this result given without proof). As common for plane duality, we will use the plane dual operator * to switch between primal and dual objects (also on sets of objects).

Extend the three half-edges of G^σ to non-crossing infinite rays and consider the planar dual of this plane graph. Since the infinite rays partition the outer face f of G into three parts, this dual contains a triangle with vertices b_1, b_2 and b_3 instead of the outer face vertex f^* such that b_i^* is not incident to r_i for every i (Fig. 2). Let the *suspended dual* G^{σ^*} of G^σ be the graph obtained from this dual by adding at each vertex of $\{b_1, b_2, b_3\}$ a half-edge pointing into the outer face.

Consider the superposition of G^σ and its suspended dual G^{σ^*} such that exactly the primal dual pairs of edges cross (here, for every $1 \le i \le 3$, the half-edge at r_i crosses the dual edge $b_{i-1}b_{i+1}$).

Definition 2. For any Schnyder wood S of G^σ, define the orientation and coloring S^* of the suspended dual G^{σ^*} as follows (Fig. 2):

(i) For every unidirected $(i-1)$-colored edge or half-edge e of G^σ, color e^* with the two colors i and $i+1$ such that e points to the right of the i-colored direction.
(ii) Vice versa, for every $i-(i+1)$-colored edge e of G^σ, $(i-1)$-color e^* unidirected such that e^* points to the right of the i-colored direction.
(iii) For every color i, make the half-edge at b_i unidirected, outgoing and i-colored.

The following lemma states that S^* is indeed a Schnyder wood of the suspended dual. The vertices b_1, b_2 and b_3 are called the *roots* of S^*.

Lemma 3 ([20][14, Prop. 3). *For every Schnyder wood S of G^σ, S^* is a Schnyder wood of G^{σ^*}.*

Since $S^{*^*} = S$, Lemma 3 gives a bijection between the Schnyder woods of G^σ and the ones of G^{σ^*}. Let the *completion* $\widetilde{G}$ of G be the plane graph obtained from the superposition of G^σ and G^{σ^*} by subdividing each pair of crossing (half-)edges with a new vertex, which we call a *crossing vertex* (Fig. 2). The completion has six half-edges pointing into its outer face.

Any Schnyder wood S of G^σ implies the following natural orientation and coloring $\widetilde{G}_S$ of its completion $\widetilde{G}$: For any edge $vw \in E(G^\sigma) \cup E(G^{\sigma^*})$, let z be the crossing vertex of G^σ that subdivides vw and consider the coloring of vw in either S or S^*. If vw is outgoing of v and i-colored, we direct $vz \in E(\widetilde{G})$ toward z and i-color it; analogously, if vw is outgoing of w and j-colored, we direct $wz \in E(\widetilde{G})$ toward z and j-color it. In the case that vw is unidirected, say without loss of generality incoming at v and i-colored, we direct $zv \in E(\widetilde{G})$ toward v and i-color it. The three half-edges of G^{σ^*} inherit the orientation and coloring of S^* for $\widetilde{G}_S$. By Definition 2, the construction of $\widetilde{G}_S$ implies immediately the following corollary.

Corollary 1. *Every crossing vertex of $\widetilde{G}_S$ has one outgoing edge and three incoming edges and the latter are colored 1, 2 and 3 in counterclockwise direction.*

Using results on orientations with prescribed outdegrees on the respective completions, Felsner and Mendez [15,21] showed that the set of Schnyder woods of a planar suspension G^σ forms a distributive lattice. The order relation of this lattice relates a Schnyder wood of G^σ to a second Schnyder wood if the former can be obtained from the latter by reversing the orientation of a directed clockwise cycle in the completion. This gives the following lemma.

Lemma 4 ([15,21])**.** *For the minimal element S of the lattice of all Schnyder woods of G^σ, $\widetilde{G}_S$ contains no clockwise directed cycle.*

We call the minimal element of the lattice of all Schnyder woods of G^σ the *minimal Schnyder wood* of G^σ.

2.3. Ordered path partitions

We denote paths as tuples of vertices such that consecutive vertices in the tuple are adjacent in the path. If a path P consists of only one vertex x, we might also write $P = x$. The concatenation of two paths P_1 and P_2 we denote by P_1P_2.

Definition 3. For any $j \in \{1,2,3\}$ and any $\{r_1,r_2,r_3\}$-internally 3-connected plane graph G, an *ordered path partition* $\mathcal{P} = (P_0,\dots,P_s)$ of G *with base-pair* (r_j,r_{j+1}) is a tuple of induced paths such that their vertex sets partition $V(G)$ and the following holds for every $i \in \{0,\dots,s-1\}$, where $V_i := \bigcup_{q=0}^{i} V(P_q)$ and the *contour* C_i is the clockwise walk from r_{j+1} to r_j on the outer face of $G[V_i]$.

(i) P_0 is the clockwise path from r_j to r_{j+1} on the outer face boundary of G, and $P_s = r_{j+2}$.
(ii) Every vertex in P_i has a neighbor in $V(G) \setminus V_i$.
(iii) C_i is a path.
(iv) Every vertex in C_i has at most one neighbor in P_{i+1}.

Remark 1. Our definition of an ordered path partition $\mathcal{P} = (P_0,\dots,P_s)$ is essentially the definition of Badent et al. [22], in which the paths P_i need to be induced (this is not explicitly stated in [22], but used in the proof of their Theorem 5).

By Definition 3(i) and (ii), G contains for every i and every vertex $v \in P_i$ a path from v to r_{j+2} that intersects V_i only in v. Since G is plane, we conclude the following.

Lemma 5. *Every path P_i of an ordered path partition is embedded into the outer face of $G[V_{i-1}]$ for every $1 \le i \le s$.*

2.3.1. Compatible ordered path partitions

We describe a connection between Schnyder woods and ordered path partitions that was first given by Badent et al. [22, Theorem 5] and then revisited by Alam et al. [23, Lemma 1].

Definition 4. Let $j \in \{1,2,3\}$ and S be any Schnyder wood of the suspension G^σ of G. As proven in [23, arXiv version, Section 2.2], the inclusion-wise maximal $j-(j+1)$-colored paths of S then form an ordered path partition of G with base pair (r_j,r_{j+1}), whose order is a linear extension of the partial order given by reachability in the acyclic graph $T_j^{-1} \cup T_{j+1}^{-1} \cup T_{j+2}$; we call this special ordered path partition *compatible* with S and denote it by $\mathcal{P}^{j,j+1}$.

For example, for the Schnyder wood given in Fig. 2, $\mathcal{P}^{2,3}$ consists of the six maximal 2-3-colored paths, of which four are single vertices. We denote each path $P_i \in \mathcal{P}^{j,j+1}$ by $P_i := (v_1^i,\dots,v_k^i)$ such that $v_1^iv_2^i$ is outgoing j-colored at v_1^i.

Let C_i be as in Definition 3. By Definition 3(iii) and Lemma 5, every path $P_i = (v_1^i,\dots,v_k^i)$ of an ordered path partition satisfying $i \in \{1,\dots,s\}$ has a neighbor $v_0^i \in C_{i-1}$ that is closest to r_{j+1} and a different neighbor $v_{k+1}^i \in C_{i-1}$ that is closest to r_j (Fig. 3). We call v_0^i the *left neighbor* of P_i, v_{k+1}^i the *right neighbor* of P_i and $P_i^e := v_0^iP_iv_{k+1}^i$ the *extension* of P_i; we omit superscripts if these are clear from the context. For $0 < i \le s$, let the path P_i *cover* an edge e or a vertex x if e or x is contained in C_{i-1}, but not in C_i, respectively.

Lemma 6 ([11])**.** *Every path $P_i \neq P_0$ of a compatible ordered path partition $\mathcal{P}^{j,j+1}$ satisfies the following (Fig. 3):*

(i) Every neighbor of P_i that is in V_{i-1} is contained in the path of C_{i-1} between v_0^i and v_{k+1}^i.
(ii) $v_0^iv_1^i$ and $v_k^iv_{k+1}^i$ are edges of $G[V_i]$.
(iii) $v_0^iv_1^i$ is $(j+1)$-colored outgoing at v_1^i and $v_k^iv_{k+1}^i$ is j-colored outgoing at v_k^i.
(iv) Every edge v_l^ix incident to P_i and V_{i-1} except for $v_0^iv_1^i$ and $v_k^iv_{k+1}^i$ is unidirected toward P_i, $(j+2)$-colored and satisfies $x \notin \{v_0^i,v_{k+1}^i\}$.

3. Bound on vertices of degree 4

Let G be a $\{r_1,r_2,r_3\}$-internally 3-connected plane graph of order n with a dual graph of order n' and a minimal Schnyder wood S. Define $B_{j,j+1}$ ($B'_{j,j+1}$) to be the number of singletons in $\mathcal{P}^{j,j+1}$ of the primal (dual) graph and $A_{j,j+1}$ ($A'_{j,j+1}$) to be the number of paths in $\mathcal{P}^{j,j+1}$ of the primal (dual) graph. We show that we can find a tree and co-tree with maximum degree four such that the number of

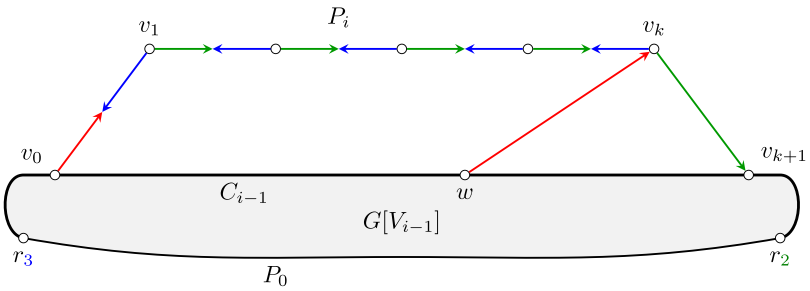


Fig. 3. Illustration for Lemmas 6 and 7.

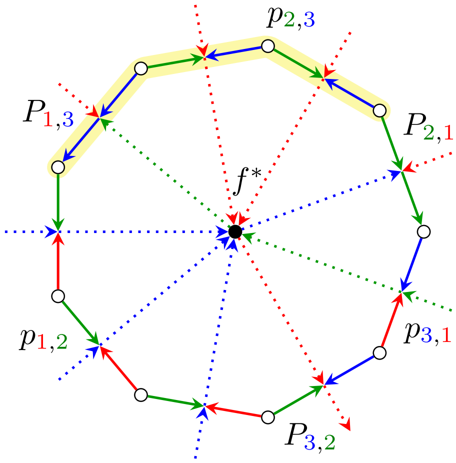


Fig. 4. Illustration for Lemma 8. A face f, the paths on its boundary and the dual edges incident to f^*. P_f is marked in yellow. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

vertices with degree four in the tree is at most $\min\{B'_{2,3}, n'+2-A'_{3,1}, n'+2-A'_{1,2}\}-1$ and the number of vertices with degree four in the co-tree is at most $\min\{B_{2,3}, n-A_{3,1}, n-A_{1,2}\}-1$.

Remember that for a spanning subgraph T of a plane graph G, $\neg T^*$ is the spanning subgraph $(V^*, (E(G)-E(T))^*)$ of G^*. As stated in the introduction, $\neg T^*$ is a spanning tree if T is one and in that case called a co-tree.

We start with two lemmas on the structure of Schnyder woods. Then, we define our candidate graphs H and H' that have the same structural properties. We show that they both have maximum degree at most 3 and that for every edge of G either the edge itself is in H or its dual is in H'. We observe that for a cycle C in H the path with the highest index in $\mathcal{P}^{2,3}$ that contains a vertex of C needs to be a singleton. This is the key observation that leads to the upper bound on the number of degree-4-vertices. Then, we eventually prove the main theorem. The proof of the main theorem uses similar tools as presented in [11]. However, our candidate graph H which is different from the one used in [11] and the aforementioned observation additionally yield the upper bound on the number of vertices of degree 4.

Lemma 7 ([11]). *Let G be a σ-internally 3-connected plane graph, S be the minimal Schnyder wood of G^σ and $\mathcal{P}^{2,3}=(P_0,\dots,P_s)$ be the ordered path partition that is compatible with S. Let $P_i := (v_1,\dots,v_k) \neq P_0$ be a path of $\mathcal{P}^{2,3}$ and v_0 and v_{k+1} be its left and right neighbor. Then, every edge $v_l w \notin \{v_0v_1, v_kv_{k+1}\}$ with $v_l \in P_i$ and $w \in V_{i-1}$ is unidirected, 1-colored and incoming at v_k and $w \notin \{v_0, v_{k+1}\}$ Fig. 3.*

Lemma 8 (Di Battista et al[17].). *The boundary of every inner face of G can be partitioned into six paths $P_{1,3}$, $p_{2,3}$, $P_{2,1}$, $p_{3,1}$, $P_{3,2}$ and $p_{1,2}$ which appear in that clockwise order. For those paths the following holds (Fig. 4).*

(i) $P_{i,j}$ consists of one edge which is either unidirected i-colored, unidirected j-colored or $i-j$-colored. Color i is directed in clockwise direction and color j in counterclockwise direction around f.
(ii) $p_{i,j}$ consists of a possibly empty sequence of $i-j$-colored edges such that color i is directed clockwise around f.

Definition 5. Let f be an inner face. Define $P_f = (x_1,\dots,x_l)$ to be the path consisting of the edges on the boundary of f that are 2-3-colored or unidirected 3-colored such that color 3 is directed counterclockwise around f. By Lemma 8, P_f is indeed a path. It consists of $p_{2,3}$ and possibly $P_{1,3}$ (Fig. 4). Let P_f be such that color 3 is directed from x_1 to x_l. For a vertex s, let $\mathrm{cf}(s)$ be the neighbor such that $\mathrm{cf}(s)s$ is the clockwise first incoming 1-colored edge at s.

Define H to be the subgraph of G with vertex set $V(G)$ and the edge set given by

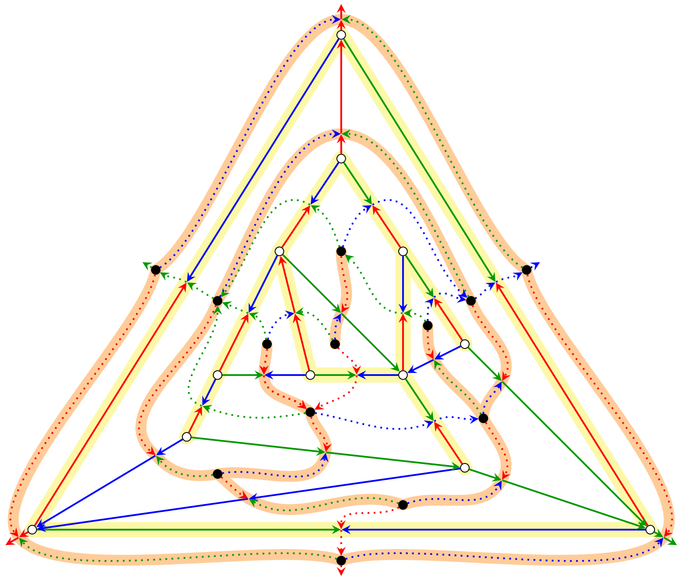


Fig. 5. Illustration for the definition of H (depicted in yellow) and H' (depicted in orange). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

(i) the 3-1-colored edges,
(ii) for every vertex s, the edge $\mathrm{cf}(s)s$,
(iii) for every inner face f, all edges of $P_f = (x_1, \ldots, x_l)$ except for $x_{l-1}x_l$,
(iv) the 2-3-colored edges on the outer face.

Define H' the same way for $G^{\sigma*}$. See Fig. 5 for illustration.

Observe that for each edge e added by Condition (iii) there exists an inner face such that e is in the 2-3-colored path $p_{2,3}$ of that face (Fig. 4). Hence, those edges are all 2-3-colored.

Lemma 9. *For an edge $e \in E(G)$, we have that $e \in E(H)$ if and only if $e^* \notin E(H')$. And thus, $H' = \neg H^* \cup \{b_1b_2, b_2b_3, b_3b_1\}$.*

Proof. In order to prove the claim, we consider all possible colorings of an edge e of G. We remind the reader of Corollary 1 which yields the relation between the coloring of an edge and its dual.

Case 1: e is 3-1-colored.
Then, e^* is unidirected 2-colored. Hence, $e \in E(H)$ by Definition 5(i), and $e^* \notin E(H')$.

Case 2: e is 1-2-colored.
Let e be incoming 1-colored at vertex s. Then, e is the first incoming 1-colored edge $\mathrm{cf}(s)s$ at s in clockwise direction by Definition 1(iii). Thus, $e \in E(H)$ by Definition 5(ii). The edge e^* is unidirected 3-colored and hence $e^* \notin E(H')$.

Case 3: e is 2-3-colored and e is not on the boundary of the outer face.
Then, e is on a path $P_f = (x_1, \ldots, x_l)$ for some face f. If $e = x_{l-1}x_l$, then $e \notin E(H)$ and $e^* = \mathrm{cf}(f^*)f^*$ by Lemma 8 and Corollary 1. Hence, $e^* \in E(H')$ by Definition 5(ii). If $e \neq x_{l-1}x_l$, then $e \in E(H)$ by Definition 5(iii), and e^* is incoming 1-colored at f^*, but $e^* \neq \mathrm{cf}(f^*)f^*$ (Fig. 4). Thus, $e^* \notin E(H')$.

Case 4: e is 2-3-colored and on the boundary of the outer face.
Then, $e \in E(H)$ by Definition 5(iv). Observe that e^* is incoming 1-colored at b_1. As b_1b_2 is also incoming 1-colored at b_1 and appears clockwise before e^*, we have that $e^* \notin E(H')$.

Case 5: e is unidirected 2-colored.
Then, e^* is 3-1-colored. Hence, $e \notin E(H)$ and $e^* \in E(H')$ by Definition 5(i).

Case 6: e is unidirected 3-colored.
Then, e^* is 1-2-colored. As in Case 2, we observe that $e \notin E(H)$ and $e^* \in E(H')$.

Case 7: $e = uv$ is unidirected 1-colored incoming at v.
Let $P_{v^*} = (x_1, \ldots, x_l)$. Assume that $e = \mathrm{cf}(v)v$. Then, $e^* = x_{l-1}x_l$. Hence, $e^* \notin E(H')$ and $e \in E(H)$ by Definition 5(ii). If $e \neq \mathrm{cf}(v)v$, then $e^* \in P_{v^*}$ and $e^* \neq x_{l-1}x_l$. We obtain that $e^* \in E(H')$ and $e \notin E(H)$. ☐

Lemma 10. *H and H' both have maximum degree at most 3.*

Proof. We show that H has maximum degree at most 3. The arguments work similarly for H'. Consider a vertex $v \in V(H)$.

Assume that v is incident to a 2-3-colored edge $e \in E(H)$ that is incoming 3-colored at v. Then, either Definition 5(iii) or (iv) applies to e. We give a short argument that in both cases there is no edge in the clockwise sector around v between e and the outgoing

3-colored edge. If Definition 5(iv) applies to e, then v is on the clockwise path from r_2 to r_3 on the boundary of the outer face, and the claim obviously holds.

Otherwise, Definition 5(iii) applies to e. Assume, for the sake of contradiction, that there is an edge in the clockwise sector around v between e and the outgoing 3-colored edge. Let e' be the clockwise first such edge. By Definition 1(iii), e' is unidirected 1-colored and incoming at v. e and e' are on a common face f. The path $P_f = (x_1, \dots, x_l)$ contains e. Since e' is not outgoing 3-colored at v, we obtain that $x_{l-1}x_l = e$ and thus, $e \notin E(H)$, a contradiction. Hence, there is no edge in the clockwise sector around v between e and the outgoing 3-colored edge.

By Definition 1(iii), the unidirected incoming 1-colored edges occur in the clockwise sector around v between e and the outgoing 3-colored edge. Thus, there is no unidirected incoming 1-colored edge at v. Hence, by Definition 5, the outgoing 1-colored edge and the outgoing 3-colored edge at v are the only additional edges incident to v that might be in $E(H)$. This yields that $\deg_H(v) \leq 3$.

Assume that v is not incident to a 2-3-colored edge $e \in E(H)$ that is incoming 3-colored at v. Then, by Definition 5, there are at most three edges incident to v that might be in $E(H)$, namely $\mathrm{cf}(v)v$, the outgoing 3-colored and the outgoing 1-colored edge. Thus, we have $\deg_H(v) \leq 3$. □

Definition 6. Let C be a cycle in H. Let $\mathcal{P}^{2,3} = (P_0, \dots, P_s)$ be the compatible ordered path partition of S. Let P be the path of maximal length in C such that $P \subseteq P_M$ with $M := \max\{i \mid P_i \cap V(C) \neq \emptyset\}$. We call P the *index-maximal subpath* of C.

Lemma 11. *Let $P_i = (v_1, \dots, v_k) \in \mathcal{P}^{2,3}$ be a path containing an index-maximal subpath P of a cycle C in H. Then, P_i is a singleton, the edge from P_i to its left neighbor v_0 is 3-1-colored and in C and the other edge in C incident to v_1 is $\mathrm{cf}(v_1)v_1$. The same holds for H'.*

Proof. We show the statement for H. The same arguments work also for H'. Assume, for the sake of contradiction, that P_i is not a singleton, i.e., $k \geq 2$. Remember that $V_i := \bigcup_{q=0}^{i} V(P_q)$. Since P_i contains the index-maximal subpath of C, $C \subseteq V_i$. By Lemma 7, the edges that connect P_i with vertices of V_{i-1} are v_0v_1 and edges with v_k as an endpoint. Hence, we either have $P = P_i$ or $P = v_k$.

If $P = v_k$, then there are two edges $e, e' \in C$ in the clockwise sector between v_kv_{k+1} and $v_{k-1}v_k$ around v_k. This sector includes v_kv_{k+1} and excludes $v_{k-1}v_k$. Observe that, by Definition 1(iii), those edges are the incoming 1-colored edges at v_k and the outgoing 2-colored edge v_kv_{k+1}. By Definition 5, of the incoming 1-colored edges, only the in clockwise order first edge $\mathrm{cf}(v_k)v_k$ is in $E(H)$. Assume that w.l.o.g. $e = \mathrm{cf}(v_k)v_k$. As $e \neq e'$, we obtain that $e' = v_kv_{k+1}$ and e' is not 1-2-colored. And since $v_{k+1} \notin P_i$, v_kv_{k+1} is also not 2-3-colored. Hence, v_kv_{k+1} is unidirected 2-colored and in H, contradicting the definition of H. Thus, we obtain that $P = P_i$.

By Lemma 7, the 3-colored outgoing edge v_0v_1 at v_1 is the only edge incident to v_1 that has an endpoint in V_{i-1}. Thus, v_0v_1 needs to be in H. By Definition 5, v_0v_1 is 3-1-colored. Then, by Definition 5(iii), $v_1v_2 \notin E(H)$, a contradiction.

This yields that P_i is a singleton. As above, we obtain that v_0v_1 needs to be 3-1-colored. By Definition 5, the other edge incident to v_1 in C is $\mathrm{cf}(v_1)v_1$. □

Definition 7. Call a singleton in $\mathcal{P}^{2,3}$ a *1-2-singleton* if its outgoing 2-colored edge is 1-2-colored and its outgoing 3-colored edge is 3-1-colored. And call a singleton in $\mathcal{P}^{2,3}$ a *2-singleton* if its outgoing 2-colored edge is unidirected and its outgoing 3-colored edge is 3-1-colored.

Observe that, by Lemma 11, index-maximal subpaths (with respect to $\mathcal{P}^{2,3}$) are either 1-2-singletons or 2-singletons. And for a 1-2-singleton s, the outgoing 2-colored edge and $\mathrm{cf}(s)s$ coincide. Recall that $B_{j,j+1}$ ($B'_{j,j+1}$) is the number of singletons in $\mathcal{P}^{j,j+1}$ of the primal (dual) graph and $A_{j,j+1}$ ($A'_{j,j+1}$) is the number of paths in $\mathcal{P}^{j,j+1}$ of the primal (dual) graph with respect to S (S^*).

Theorem 1. *Let G be a $\{r_1, r_2, r_3\}$-internally 3-connected plane graph of order n with a dual graph of order n' and a minimal Schnyder wood S of G^σ. There is a 4-tree T in G such that $\neg T^*$ is a 4-tree. Also, the number of degree-4-vertices in T is at most*

$$\min\{B'_{2,3}, n' + 2 - A'_{3,1}, n' + 2 - A'_{1,2}\} - 1$$

and the number of degree-4-vertices in $\neg T^*$ is at most

$$\min\{B_{2,3}, n - A_{3,1}, n - A_{1,2}\} - 1.$$

Proof. By Lemma 4, the completion $\widetilde{G}_S$ of G contains no clockwise directed cycle. Since $\widetilde{G}_S$ contains the completion of the suspended dual G^{σ^*} apart from its three outer vertices which do not affect clockwise cycles, S^* is a minimal Schnyder wood of G^{σ^*}.

Let H and H' be as defined in Definition 5. Recall that, by Lemma 9, $e \in E(H)$ if and only if $e^* \notin E(H')$, and thus $H' = \neg H^* \cup \{b_1b_2, b_2b_3, b_3b_1\}$. As b_1b_2, b_2b_3 and b_3b_1 are not in G^*, they do not affect our desired trees. By Lemma 10, H and H' both have maximum degree at most 3. Observe that H and H' might both have cycles and are not necessarily connected (Fig. 5).

We will therefore iteratively identify edges of cycles of H such that $\neg H^* \cup \{b_1b_2, b_2b_3, b_3b_1\}$ still has maximum degree at most 4 when those edges are deleted in H. In order to do this, we iteratively define the set of edges $D \subseteq E(H)$ that is deleted from H. Then, we use the exact same arguments in order to define the set of edges D' that is deleted from H'. We start with $D = D' = \emptyset$.

Let $\mathcal{P}^{2,3} = (P_0, \dots, P_s)$ be the compatible ordered path partition formed by the maximal 2-3-colored paths. We will consider (and formally define in the following) paths that are the first (i.e. index-minimal) path covering an index-maximal subpath. Denote by $\mathcal{P}_{\max}$ the set of all index-maximal subpaths. For a path $P \in \mathcal{P}_{\max} \setminus \{P_s\}$ refer to P_L with $L = \min\{i \mid P_i$ covers an edge of the extension of $P\}$ as the *minimal-covering path* of P. Denote by $\mathcal{P}_{cover}$ the set of the minimal-covering paths of the paths of $\mathcal{P}_{\max} \setminus \{P_s\}$.

Observe that $P_s = r_1$ is the index-maximal subpath of the outer face boundary and a 1-2-singleton. There is no path in $\mathcal{P}^{2,3}$ that covers an edge of P_s. Hence, in order to destroy the outer face cycle, we add the outgoing 3-colored edge of r_1 to D.

Next, we process the paths of $\mathcal{P}_{cover}$ in reverse order of $\mathcal{P}^{2,3}$, i.e., from highest to lowest index. Let $P_c = (v_1, \dots, v_k) \in \mathcal{P}_{cover}$, $c \in \{1, \dots, s\}$ be the path under consideration. Let $s_1, \dots, s_l$ be the index-maximal subpaths for which P_c is the minimal-covering path,

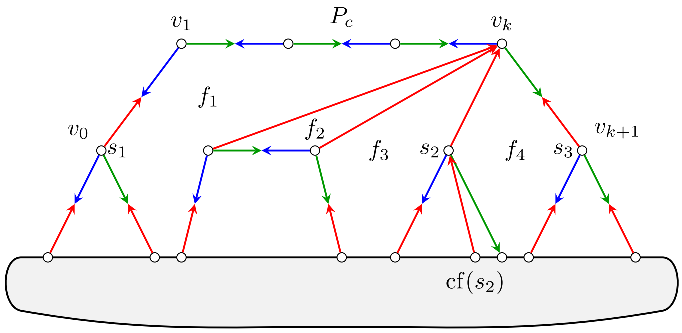


Fig. 6. Illustration for some of the definitions used in the proof of Theorem 1.

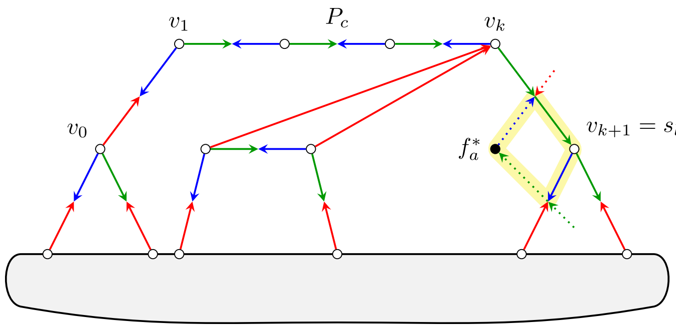


Fig. 7. Illustration for the proof of Theorem 1. If $v_k v_{k+1}$ is 2-colored, then $\widetilde{G}_S$ contains a clockwise cycle (depicted in yellow). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

ordered clockwise around the outer face of $G[V_{c-1}]$ (Fig. 6). Let $f_1, \dots, f_a$ be the faces incident to v_k in counterclockwise order from the outgoing 3-colored edge to the outgoing 2-colored edge. We say that $f_1, \dots, f_a$ are *below* P_c and *above* $s_1, \dots, s_l$.

Assume $v_{k+1} = s_l$. We give a short argument showing that in this case $v_k v_{k+1}$ is 1-2-colored. For the sake of contradiction, assume that $v_k v_{k+1}$ is 2-colored. Then, by Corollary 1, $(v_k v_{k+1})^*$ is 3-1-colored. As s_l is an index-maximal subpath the outgoing 3-colored edge at s_l is 3-1-colored. Hence, $\widetilde{G}_S$ contains the clockwise cycle in Fig. 7, which contradicts the assumption that S is the minimal Schnyder wood. We obtain that $v_k v_{k+1}$ is 1-2-colored in this case.

Remember that, by Lemma 11, $\mathrm{cf}(s)s$ and the outgoing 3-colored edge are the only edges in H that join s with vertices of V_{i-1}.

Now, we select for each of the singletons $s \in \{s_1, \dots, s_l\}$ either the outgoing 3-colored edge or $\mathrm{cf}(s)s$ and add it to D. Thus, after having processed every path in $\mathcal{P}_{cover}$, a cycle in H does not exist in $H - D$ anymore. We aim for selecting those edges that have the smallest possible impact on the maximum degree of the dual graph. Hence, for a 2-singleton s we always choose the edge $\mathrm{cf}(s)s$. Deleting this specific edge does not increase the degree of any face below P_c (Fig. 8). In detail we distinguish the following three cases.

Augmentation procedure of D for the path P_c:.

Case 1: $P_c = (v_1, \dots, v_k)$ is not an index-maximal subpath.

For every singleton $s \in \{s_1, \dots, s_l\} \setminus \{v_{k+1}\}$, we add $\mathrm{cf}(s)s$ to D. If $v_{k+1} = s_l$ is a 2-singleton, we add $\mathrm{cf}(s_l)s_l$ to D. Otherwise, we add its outgoing 3-colored edge to D (Fig. 8).

Case 2: $P_c = v_1$ is an index-maximal subpath and a 1-2-singleton. Then, we already have either $v_0 v_1 \in D$ or $v_1 v_2 \in D$.

Case 2.1: $v_0 v_1 \in D$.

We proceed as in Case 1.

Case 2.2: $v_1 v_2 \in D$.

For every singleton $s \in \{s_1, \dots, s_l\}$ that is a 2-singleton, we add $\mathrm{cf}(s)s$ to D. For every singleton $s \in \{s_1, \dots, s_l\} \setminus \{v_0\}$ that is a 1-2-singleton, we add the outgoing 3-colored edge to D. If $v_0 = s_1$ is a 1-2-singleton, then add its outgoing 2-colored edge to D (Fig. 9).

Case 3: $P_c = v_1$ is an index-maximal subpath and a 2-singleton.

Then, we already added $\mathrm{cf}(v_1)v_1$ to D. For every singleton $s \in \{s_1, \dots, s_l\}$, we add $\mathrm{cf}(s)s$ to D. Observe that $v_1 v_2$ is unidirected 2-colored and hence $v_2 \neq s_l$ (Fig. 10).

Now, we need to show that the maximum degree of $\neg H^* + D^*$ is at most 4. We prove that, after having processed P_c, no further boundary edge of any $f \in \{f_1, \dots, f_a\}$ is added to D: Assume to the contrary that there is a face $f \in \{f_1, \dots, f_a\}$ and an edge e on the boundary of f such that e is not in D after having processed P_c but will be added later. Observe that e is not a unidirected 1-colored

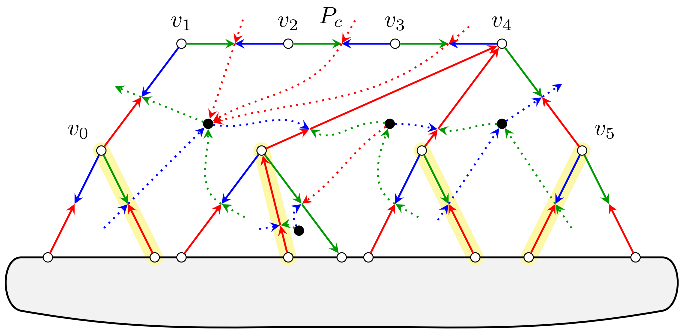


Fig. 8. The situation if P_c is a path. The edges that we add to D are marked in yellow. In the depicted situation, v_1v_2 is not in H. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

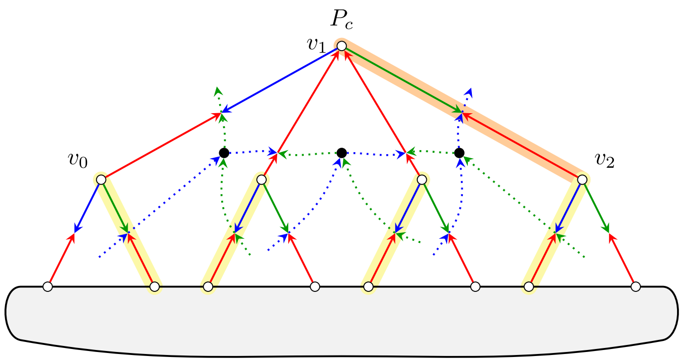


Fig. 9. The situation in Case 2.2. The edge v_1v_2 is marked in orange and in D before we consider P_c. The edges that we then add to D are marked in yellow. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

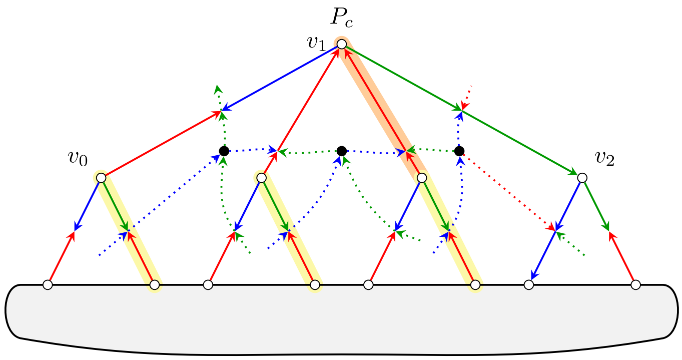


Fig. 10. The situation in Case 3. The edge $\mathrm{cf}(v_1)v_1$ is marked in orange and in D before we consider P_c. The edges that we then add to D are marked in yellow. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

edge by Lemma 8. Hence, e is in the extension of some path $P_i \in \mathcal{P}^{2,3}$. Then, the minimal-covering path $P_{c'} \in \mathcal{P}^{2,3}$ of P_i needs to have lower index than P_c, i.e., $c' < c$. As e is covered by P_c, it is not covered by the minimal-covering path of P_i. Hence, e will not be added to D, a contradiction.

First, consider the case $a > 1$, i.e., there are at least two faces below P_c. By Definition 3(ii), the boundary of every f_j with $j \in \{1, \dots, a\}$ contains at most two edges that are in the union of the extensions of singletons in $\{s_1, \dots, s_l\}$. In Case 1 and 2, for every $j \in \{2, \dots, a-1\}$, we add at most one edge of the boundary of f_j to D. This implies that $\deg_{\neg H^*+D^*}(f_j^*) \le 4$ for every $j \in \{2, \dots, a-1\}$ (Figs. 6 and 9). The same holds for every $j \in \{1, \dots, a-2\}$ in Case 3 (Fig. 10).

Let us now consider f_1^* in the case $a > 1$. In Case 1, we add at most one edge of the boundary of f_1 to D (Fig. 8). Hence, in that case $\deg_{\neg H^*+D^*}(f_1^*) \le 4$. In the other two cases, P_c is either a 2-singleton or a 1-2-singleton. Thus, v_1v_0 is 3-1-colored and, by Corollary 1, $(v_1v_0)^*$ is unidirected 2-colored and outgoing at f_1^*. The outgoing 3-colored edge at f_1^* is 2-3-colored since its primal edge is unidirected 1-colored and incoming in v_1 (Fig. 9). Also, f_1^* does not have any incoming 1-colored edges. This implies that, by Definition 5, the edges incident to f_1^* that might be in $\neg H^*$ are only the outgoing 3-colored edge and the outgoing 1-colored edge.

Hence, $\deg_{\neg H^*}(f_1^*) \le 2$. In Case 2.1, 2.2 and 3, if $v_0 = s_1$ and s_1 is a 1-2-singleton, then the edge from s_1 to its right neighbor is in D and on the boundary of f_1. Now, we argue that in each of those cases at most one more edge on the boundary of f_1 might be in D. In Case 2.1, v_0v_1 is the only additional edge on the boundary of f_1 that is in D. In Case 2.2, if s_2 is on the boundary of f_1, then the edge from s_2 to its left neighbor is in D (Fig. 9). This is the only additional edge of the boundary of f_1 that might be in D in Case 2.2. Finally, in Case 3 (here $P_c = v_1$ is a 2-singleton), if $\mathrm{cf}(v_1)$ is on the boundary of f_1, then $\mathrm{cf}(v_1)v_1$ is on the boundary of f_1 and in D. Again, this is the only additional edge of the boundary of f_1 that might be in D in Case 3. Hence, in any of those cases $\deg_{\neg H^*+D^*}(f_1^*) \le 4$.

Consider f_a^* in the case $a > 1$. As $a > 1$, the outgoing 2-colored edge at f_a^* is 2-3-colored. Assume that v_kv_{k+1} is 1-2-colored. Then, $(v_kv_{k+1})^*$ is unidirected 3-colored and outgoing at f_a^* by Corollary 1. This implies, by Definition 5(iii) that $(v_kv_{k+1})^*$ is not in H and the outgoing 2-colored edge at f_a^* is in H. Also, there is no incoming 1-colored edge at f_a^*. Thus, $\deg_{\neg H^*}(f_a^*) \le 2$. Since in any case (Case 3 cannot occur here) we add at most two edges of the boundary of f_a to D, we obtain that $\deg_{\neg H^*+D^*}(f_a^*) \le 4$ (Figs. 8 and 9). So assume that v_kv_{k+1} is 2-colored. By Definition 5, v_kv_{k+1} is not in H and hence, by Lemma 9, $(v_kv_{k+1})^* \in E(\neg H^*)$. As above there is no additional incoming 1-colored edge at f_a^*. The outgoing 2-colored edge at f_a^* is 2-3-colored since it is the dual of a unidirected 1-colored edge. But this unidirected 1-colored edge is the clockwise first incoming 1-colored edge at v_k and hence in H. Thus, by Lemma 9, the outgoing 2-colored edge at f_a^* is not in $\neg H^*$. The outgoing 1-colored edge at f_a^* is potentially in $\neg H^*$. This yields that, $\deg_{\neg H^*}(f_a^*) \le 2$ and since we add at most two edges to D that are on the boundary of f_a, we have that $\deg_{\neg H^*+D^*}(f_a^*) \le 4$.

Consider f_{a-1}^* in Case 3. The edges incident to f_{a-1}^* that could possibly be in $\neg H^* + D^*$ are an incoming 3-colored edge and its three outgoing edges (Fig. 10). Thus, $\deg_{\neg H^*+D^*}(f_{a-1}^*) \le 4$.

In the remaining case $a = 1$, there is exactly one face below P_c. If P_c is not in $\mathcal{P}_{\max}$, we use exactly the same arguments that we used to show that $\deg_{\neg H^*+D^*}(f_a^*) \le 4$ for $a \ne 1$. If P_c is an index-maximal subpath, then P_c is a 1-2-singleton, as it cannot be a 2-singleton. By Corollary 1, the outgoing 2-colored and the outgoing 3-colored edge at f_1^* are unidirected. Also, f_1^* does not have any 1-colored incoming edges by Definition 1(iii). Thus, only the outgoing 1-colored edge might be in $\neg H^*$ and we have $\deg_{\neg H^*}(f_1^*) \le 1$. We add at most three edges of the boundary of f_1 to D. Those edges are; an edge of the extension of P_c, the outgoing 2-colored edge of v_0 and the outgoing 3-colored edge of v_2. And we obtain that $\deg_{\neg H^*+D^*}(f_1^*) \le 4$.

We are left to show that a vertex f'^* of $\neg H^* + D^*$ such that f' is not below a path of $\mathcal{P}_{cover}$ has degree at most 3 in $\neg H^* + D^*$. If f'^* is not incident to an edge of D^*, then we have that $\deg_{\neg H^*+D^*}(f'^*) = \deg_{\neg H^*}(f'^*) \le 3$. If f'^* is incident to an edge of D^*, then f'^* is below a path P of $\mathcal{P}_{\max}$. Such a path is either a 1-2-singleton or a 2-singleton. Hence, f'^* does not have unidirected incoming 1-colored edges. We have that $P = u_1$ for a vertex $u_1 \in V(G)$. Assume that P is a 1-2-singleton. Then, either $u_0u_1 \in D$ or $u_1u_2 \in D$. And thus, f'^* is either incident to $(u_0u_1)^*$ or $(u_1u_2)^*$ in $\neg H^* + D^*$. If f'^* is incident to $(u_0u_1)^*$ ($(u_1u_2)^*$) in $\neg H^* + D^*$, then the only edges incident to f'^* that might be in $\neg H^*$ are its outgoing 2-colored (3-colored) edge and its outgoing 1-colored edge, i.e., $\deg_{\neg H^*}(f'^*) \le 2$ and thus $\deg_{\neg H^*+D^*}(f'^*) \le 3$. So assume that P is a 2-singleton. Then, f'^* is incident to the dual of the clockwise first incoming 1-colored edge of u_1. This dual is 2-3-colored. As f'^* does not have unidirected incoming 1-colored edges, the edges incident to f'^* that are potentially in $\neg H^* + D^*$ are its outgoing 2-colored, its outgoing 3-colored and its outgoing 1-colored edge (Fig. 10, but ignore the edges marked in yellow). We obtain that $\deg_{\neg H^*+D^*}(f'^*) \le 3$.

We now show that we can assign degree-4-vertices of $\neg H^* + D^*$ injectively to 1-2-singletons of $\mathcal{P}^{2,3}$ of G. This we later need in order to prove the desired upper bounds on the number of vertices of degree 4. Consider the arguments that show that $\neg H^* + D^*$ has maximum degree at most 4. They also yield that every degree-4-vertex f^* of $\neg H^* + D^*$ such that f is below a path of $\mathcal{P}_{cover}$ has at least one 1-2-singleton x on its boundary such that f^* is above x and either the outgoing 2-colored edge at x or the outgoing 3-colored edge at x is on the boundary of f and in D.

This yields that each degree-4-vertex f^* in $\neg H^* + D^*$ has at least one 1-2-singleton x of $\mathcal{P}^{2,3}$ of G on its boundary such that f^* is above x and either the outgoing 2-colored edge at x or the outgoing 3-colored edge at x is on the boundary of f and in D. We assign each degree-4-vertex to such a 1-2-singleton. Since we never add both the outgoing 2-colored and the outgoing 3-colored edge of a 1-2-singleton to D, this assignment is injective. Also, we know that r_1 is a 1-2-singleton but as there is no path in $\mathcal{P}_{cover}$ covering r_1, no degree-4-vertex is assigned to r_1. Thus, we obtain that the number of degree-4-vertices in $\neg H^* + D^*$ is at most the number of 1-2-singletons minus one. As every 1-2-singleton has a 3-1-colored and a 1-2-colored edge, we obtain

$$\begin{aligned}\#1-2-\text{singletons}-1 &\le \min\{B_{2,3}, \#3\text{-1-colored edges}, \#1\text{-2-colored edges}\}-1\\ &\le \min\{B_{2,3}, n-A_{3,1}, n-A_{1,2}\}-1.\end{aligned}$$

So far, we showed that $H - D$ is acyclic, $\neg H^* + D^*$ has maximum degree at most 4 and that our desired upper bound on the number of degree-4-vertices holds for $\neg H^* + D^*$. We now apply the same arguments that we used for H to $H' = \neg H^* \cup \{b_1b_2, b_2b_3, b_3b_1\}$ and obtain D'. Hence, we have that $H' - D'$ is acyclic and $H + D'^* \setminus \{b_1b_2, b_2b_3, b_3b_1\}^*$ has maximum degree at most 4. Since G^* and $G^{\sigma*}$ differ on the outer face we obtain the following bound on the number of degree-4-vertices in $H + D'^* \setminus \{b_1b_2, b_2b_3, b_3b_1\}^*$ of

$$\begin{aligned}\#1\text{-2-singletons}-1 &\le \min\{B'_{2,3}, \#3\text{-1-colored edges}, \#1\text{-2-colored edges}\}-1\\ &\le \min\{B'_{2,3}, n'+2-A'_{3,1}, n'+2-A'_{1,2}\}-1.\end{aligned}$$

In this formula, we refer to the number of 1-2-singletons and the number of 3-1-colored and 1-2-colored edges of the ordered path partition of G^{σ^*} given by the maximal 2-3-colored paths.

The edges b_1b_2, b_2b_3 and b_3b_1 are not in G^* and there is only one edge on the boundary of the outer face of G that is also in D. We may thus ignore b_1b_2, b_2b_3 and b_3b_1 in the following and freely switch from $\neg H^* \cup \{b_1b_2, b_2b_3, b_3b_1\}$ to $\neg H^*$. Hence, we also remove any of the edges b_1b_2, b_2b_3, b_3b_1 from D'.

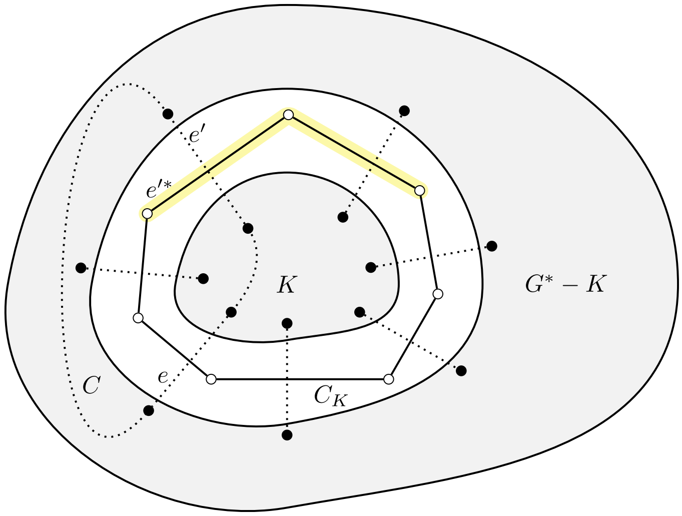


Fig. 11. Illustration for the proof of Theorem 1. The clockwise first incoming 1-colored and the outgoing 3-colored edge of the singleton P^{C_K} are highlighted in yellow. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Then, the graphs $\neg H^* - D' + D^*$ and $H - D + D'^*$ have maximum degree at most 4 and by construction $\neg H^* - D' + D^* = \neg(H - D + D'^*)^*$. An edge set $E \subseteq E(G)$ is the edge set of a cycle in G if and only if the edge set E^* is a minimal cut in G^* [24, Prop. 4.6.1]. Hence, in order to show that $\neg H^* - D' + D^*$ and $H - D + D'^*$ are both trees it suffices to show that they are both acyclic. We show that $\neg H^* - D' + D^*$ is acyclic. The same arguments might then be applied to $H - D + D'^*$.

For the sake of contradiction, assume that there is a cycle C in $\neg H^* - D' + D^*$. Remember that each index-maximal subpath in $\mathcal{P}_{\max}$ is a singleton. We either pick its outgoing 3-colored or its in clockwise direction first incoming 1-colored edge and add it to D. This will eventually lead to a contradiction. By construction, every cycle in $\neg H^*$ has at least one edge that is also in D'. Hence, C has at least one edge of D^*. Since every edge of D is in a cycle of H, by [24, Prop. 4.6.1], every edge in D^* joins two vertices of two different connected components of $\neg H^*$.

For a connected component K of $\neg H^*$ let $E_K \subseteq E(G^*)$ be the minimal edge cut separating K and $G^* - K$. Let C_K be the cycle of G with $E(C_K) = E_K^*$ and let $P^{C_K} = P_i \in \mathcal{P}^{2,3}$ be the index-maximal subpath of C_K (Fig. 11). Choose K such that K contains a vertex of C and $P^{C_K} = P_i$ has smallest index. Since C is a cycle and intersects at least two connected components of $\neg H^*$, there are two edges $e, e' \in E_K$ that are also in C. Observe that these edges need to be in D^*.

Then, either e^* or e'^* is not the clockwise first incoming 1-colored or the outgoing 3-colored edge of P^{C_K}. Assume w.l.o.g. that e'^* is the clockwise first incoming 1-colored or the outgoing 3-colored edge of P^{C_K}. Let $P' = P_j \in \mathcal{P}^{2,3}$ for some $j \in \{1, \dots, s\}$ be the path such that e^* is its outgoing 3-colored or its in clockwise direction first incoming 1-colored edge. Since $P^{C_K} = P_i$ is the index-maximal subpath of C_K, we have $j < i$. Hence, there exists a connected component K' of $\neg H^*$ such that K' and C have a vertex in common and $P' = P_j$ is the index-maximal subpath of the cycle $C_{K'}$ with $(E(C_{K'}))^*$ being the minimal cut separating K' and $G^* - K'$. This contradicts the definition of K. Thus, $\neg H^* - D' + D^*$ and $H - D + D'^*$ are both trees. This concludes the proof. □

Corollary 2. *The vertex r_1 is a leaf in $H - D + D'^*$. All edges on the outer face of G except for the outgoing 2-colored edge at r_1 are in $H - D + D'^*$. Hence, the dual vertex of the outer face is also a leaf in $\neg H^* - D' + D^*$. Furthermore, we have that $\deg_{H-D+D'^*}(r_2) = 2$ and $\deg_{H-D+D'^*}(r_3) \leq 3$.*

Proof. Observe that in the proof of Theorem 1, there are only 3-1-colored, 1-2-colored and unidirected 1-colored edges in D' and D. Thus, by Corollary 1, there are only unidirected 2-colored, unidirected 3-colored and 2-3-colored edges in D'^* and D^*. The only edges that are incident to r_1 in H are its outgoing 3-colored and its outgoing 2-colored edge. The outgoing 3-colored edge at r_1 is in D. Apart from those two edges, r_1 is incident to incoming unidirected 1-colored edges only. Hence, there is no edge in D'^* that is incident to r_1 and thus r_1 is a leaf in $H - D + D'^*$.

The dual edges of the incoming unidirected edges at r_3 and r_2 are all covered by the last singleton b_1 of $\mathcal{P}^{2,3}$ of G^{σ^*} (Fig. 5). Let e_3 be the dual of the counterclockwise first unidirected 3-colored incoming edge at r_3 and e_2 be the dual of the clockwise first unidirected 2-colored incoming edge at r_2. Let I_i be the set of the duals of the unidirected i-colored incoming edges at r_i, $i = 2, 3$. By Definition 5(i) and (ii), all edges in I_2 and I_3 are also in $\neg H^*$, respectively. For $e \in I_i$, $i = 2, 3$, let P_e be the path such that e belongs to the extension of P_e. Observe that for all edges $e \in (I_3 \setminus \{e_3\}) \cup (I_2 \setminus \{e_2\})$ the root b_1 covers e but is not the minimal-covering path of P_e. Hence, those edges are not added to D'. On the other hand, b_1 can be the minimal-covering path of P_{e_3} and/or P_{e_2}. Since we added $b_1 b_3$ to D', Case 2.1 of the proof of Theorem 1 applies to b_1. Thus, we do not add e_2 to D' but might add e_3. We obtain that $\deg_{H-D+D'^*}(r_2) = 2$ and $\deg_{H-D+D'^*}(r_3) \leq 3$.

By Definition 5(iv), the edges on the outer face of G are in H. We add the outgoing 3-colored edge at r_1 to D. The other edges on the outer face are not added to D because they are either not covered by a path (3-1-colored and 1-2-colored edges) or they are not incident to a singleton (2-3-colored edges) of $\mathcal{P}^{2,3}$ of G. Hence, the dual vertex of the outer face is a leaf in $\neg H^* - D' + D^*$. □

CRediT authorship contribution statement

Christian Ortlieb: Writing – review & editing, Writing – original draft, Visualization, Validation, Methodology, Investigation, Formal analysis, Data curation, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

This research is supported by the Deutsche Forschungsgemeinschaft grant SCHM 3186/2-1 (401348462) from the (DFG, German Research Foundation).

References

[1] D. Barnette, Trees in polyhedral graphs, Can. J. Math. 18 (1966) 731–736.
[2] T. Böhme, J. Harant, M. Kriesell, S. Mohr, J.M. Schmidt, Rooted minors and locally spanning subgraphs, J. Graph Theory 105 (2) (2024) 209–229. https://doi.org/10.1002/jgt.23012
[3] K. Ota, K. Ozeki, Spanning trees in 3-connected $K_{3,t}$-minor-free graphs, J. Comb. Theory Series B 102 (2012) 1179–1188.
[4] K. Ozeki, T. Yamashita, Spanning trees: a survey, Graphs Comb. 27 (2011) 1–26.
[5] D.W. Barnette, 2-connected spanning subgraphs of planar 3-connected graphs, J. Comb. Theory Series B 61 (1994) 210–216.
[6] Z. Gao, 2-connected coverings of bounded degree in 3-connected graphs, J. Graph Theory 20 (3) (1995) 327–338.
[7] T. Biedl, Trees and co-trees with bounded degrees in planar 3-connected graphs, in: 14th Scandinavian Symposium and Workshops on Algorithm Theory (SWAT'14), 2014, pp. 62–73.
[8] Z. Gao, R.B. Richter, 2-Walks in circuit graphs, J. Comb. Theory Series B 62 (2) (1994) 259–267.
[9] R. Brunet, M.N. Ellingham, Z. Gao, A. Metzlar, R.B. Richter, Spanning planar subgraphs of graphs in the torus and klein bottle, J. Comb. Theory Series B 65 (1) (1995) 7–22. https://doi.org/10.1006/jctb.1995.1041
[10] B. Grünbaum, Polytopes, graphs, and complexes, Bull. Am. Math. Soc. 76 (6) (1970) 1131–1201.
[11] C. Ortlieb, J.M. Schmidt, Toward Grünbaum's conjecture, in: 19th Scandinavian Symposium and Workshops on Algorithm Theory (SWAT 2024), 294 of *Leibniz International Proceedings in Informatics (LIPIcs)*, Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl, Germany, 2024, pp. 37:1–37:17. https://doi.org/10.4230/LIPIcs.SWAT.2024.37
[12] C. Ortlieb, Toward Grünbaum's conjecture for 4-connected graphs, in: Proceedings of the 49th International Symposium on Mathematical Foundations of Computer Science (MFCS'24), 306 of Leibniz International Proceedings in Informatics, 2024, pp. 77:1–77:13. https://doi.org/10.4230/LIPICS.MFCS.2024.77
[13] S. Felsner, Geodesic embeddings and planar graphs, Order 20 (2003) 135–150.
[14] W. Schnyder, Embedding planar graphs on the grid, in: Proceedings of the 1st Annual ACM-SIAM Symposium on Discrete Algorithms, 1990, pp. 138–148.
[15] S. Felsner, Geometric Graphs and Arrangements, Advanced Lectures in Mathematics, Vieweg+Teubner, Wiesbaden, 2004. https://doi.org/10.1007/978-3-322-80303-0
[16] S. Felsner, Convex drawings of planar graphs and the order dimension of 3-polytopes, Order 18 (1) (2001) 19–37. https://doi.org/10.1023/A:1010604726900
[17] G. Di Battista, R. Tamassia, L. Vismara, Output-sensitive reporting of disjoint paths, Algorithmica 23 (4) (1999) 302–340. https://doi.org/10.1007/PL00009264
[18] S. Felsner, Lattice structures from planar graphs, Electron. J. Comb. 11 (1) (2004) R15, 1–24.
[19] G. Kant, Drawing planar graphs using the canonical ordering, Algorithmica 16 (1) (1996) 4–32. https://doi.org/10.1007/BF02086606.
[20] G. Kant, Drawing planar graphs using the lmc-ordering, in: Proceedings of the 33rd Annual Symposium on Foundations of Computer Science (FOCS'92), 1992, pp. 101–110. https://doi.org/10.1109/SFCS.1992.267814
[21] P.O. de Mendez, Orientations Bipolaires, Ph.D. thesis, École des Hautes Études en Sciences Sociales, Paris, 1994.
[22] M. Badent, U. Brandes, S. Cornelsen, More canonical ordering, J. Graph Algorithms Appl. 15 (1) (2011) 97–126.
[23] M.J. Alam, W. Evans, S.G. Kobourov, S. Pupyrev, J. Toeniskoetter, T. Ueckerdt, Contact representations of graphs in 3D, in: Proceedings of the 14th International Symposium on Algorithms and Data Structures (WADS '15), 9214 of Lecture Notes in Computer Science, 2015, pp. 14–27. Technical Report accessible on arXiv:arxiv.org/abs/1501.00304. https://doi.org/10.1007/978-3-319-21840-3_2
[24] R. Diestel, Graph theory, Graduate texts in mathematics ; 173, Springer, Berlin, 4. ed., 2., corr. print. edition, 2012. http://swbplus.bsz-bw.de/bsz377230375cov.htm.