Toward Grünbaum's Conjecture Bounding Vertices of Degree 4*

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Abstract

Given a spanning tree T of a planar graph G, the *co-tree* of T is the spanning tree of the dual graph G^* with edge set $(E(G) - E(T))^*$. Grünbaum conjectured in 1970 that every planar 3-connected graph G contains a spanning tree T such that both T and its co-tree have maximum degree at most 3.

While Grünbaum's conjecture remains open, Schmidt and the author recently improved the upper bound on the maximum degree from 5 (Biedl 2014) to 4.

In this paper, we modify this approach taking a further step towards Grünbaum's conjecture. We again obtain a spanning tree T such that both T and its co-tree have maximum degree at most 4 and, additionally, an upper bound on the number of vertices of degree 4 of T and its co-tree.

Keywords

Planar graph, spanning tree, maximum degree, Schnyder wood

1. Introduction

Let a k*-tree* be a spanning tree whose maximum degree is at most k. In 1966, Barnette proved the fundamental theorem that every planar 3-connected graph contains a 3-tree [1]. Both assumptions of Barnette's theorem are essential in the sense that the statement fails for arbitrary non-planar graphs (as the arbitrarily high degree in any spanning tree of the graphs $K_{3,n-3}$ show) as well as for graphs that are not 3-connected (as the planar graphs $K_{2,n-2}$ show).

Since then, Barnette's theorem has been extended and generalized in several directions [2, 3, 4, 5, 6, 7, 8, 9]. Perhaps one of the most severe strengthenings is a long-standing and to the best of our knowledge still open conjecture made by Grünbaum in 1970. Since the planar dual G^* of every (simple) planar 3-connected graph G is again planar and 3-connected, G^* contains a 3-tree as well. By the well-known cut-cycle duality, any spanning tree T of G implies that also $(V^*, (E(G) - E(T))^*)$ is a spanning tree $\neg T^*$ of G^*; we call $\neg T^*$ the *co-tree* of T. Taking the best of these two worlds, Grünbaum made the following conjecture.

Conjecture (Grünbaum [10, p. 1148], 1970)**.** *Every planar 3-connected graph G contains a 3-tree T whose co-tree $\neg T^*$ is also a 3-tree.*

While Grünbaum's conjecture is to the best of our knowledge still unsolved, progress has been made by Biedl [7], who proved the existence of a 5-tree whose co-tree is a 5-tree. Exploiting insights into the structure of Schnyder woods, Schmidt and the author [11] proved the existence

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of a 4-tree whose co-tree is a 4-tree. In this paper, we additionally give upper bounds on the number of vertices of degree 4 in both the tree and the co-tree. The approach is similar to the one in [11]. We use a different candidate graph, show that it meets all necessary conditions and then apply a method similar to the one in [11]. We observe that only under specific local conditions vertices of degree 4 might arise. Thus, we are able to count them.

We fix a minimal Schnyder wood S of G. S gives rise to a Schnyder wood of the suspended dual (a graph that differs form the dual graph only on the outer face). Also, every Schnyder wood has three compatible ordered path partitions, denoted by $\mathcal{P}^{j,j+1}$, $j \in \{1, 2, 3\}$. We define and explain those concepts in Section 2.

The upper bound on the number of degree-4-vertices in T and $\neg T^*$ is then given by $\min\{B'_{2,3}, n' + 2 - A'_{3,1}, n' + 2 - A'_{1,2}\} - 1$ and $\min\{B_{2,3}, n - A_{3,1}, n - A_{1,2}\} - 1$, respectively. Here n (n') is the order of the primal (dual) graph, $B_{j,j+1}$ ($B'_{j,j+1}$) is the number of singletons in $\mathcal{P}^{j,j+1}$ of the primal (dual) graph and $A_{j,j+1}$ ($A'_{j,j+1}$) is the number of paths in $\mathcal{P}^{j,j+1}$ of the primal (dual) graph.

Our arguments work symmetrically for any choice of two colors. Thus, we obtain for example that if one of the compatible ordered path partitions of the minimal Schnyder wood of the dual graph has only one singleton (which is the minimum number of singletons), then there exists a spanning tree of maximum degree at most 3 such that its co-tree has maximum degree at most 4.

We discuss Schnyder woods, their lattice structure and ordered path partitions in Section 2 and our main result in Section 3. Due to space limitations some proofs are omitted or only sketched.

2. Schnyder Woods and Ordered Path Partitions

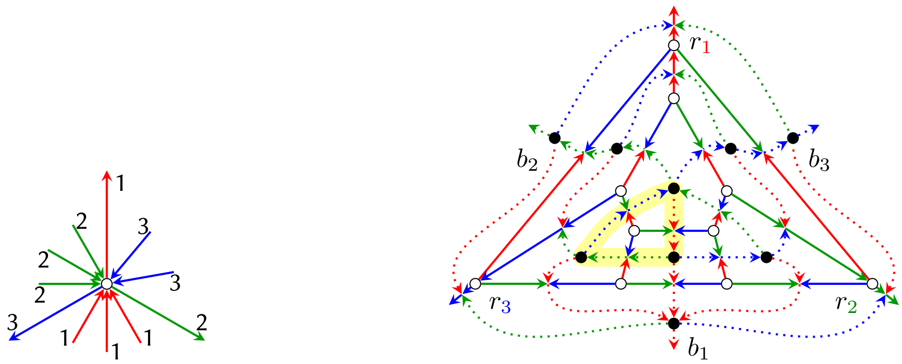
We only consider simple undirected graphs. A graph is *plane* if it is planar and embedded into the Euclidean plane without intersecting edges. The *neighborhood of a vertex set* A is the union of the neighborhoods of vertices in A. Although parts of this paper use orientation on edges, we will always let vw denote the undirected edge $\{v, w\}$.

2.1. Schnyder Woods

Let $\sigma := \{r_1, r_2, r_3\}$ be a set of three vertices of the outer face boundary of a plane graph G in clockwise order (but not necessarily consecutive). We call r_1, r_2 and r_3 *roots*. The *suspension* G^σ of G is the graph obtained from G by adding at each root of σ a half-edge pointing into the outer face. With a little abuse of notation, we define a *half-edge* as an arc starting at a vertex but with no defined end vertex. A plane graph G is σ*-internally 3-connected* if the graph obtained from the suspension G^σ of G by making the three half-edges incident to a common new vertex inside the outer face is 3-connected. The class of σ-internally 3-connected plane graphs properly contains all 3-connected plane graphs.

Definition 1 (Felsner [12]). *Let $\sigma = \{r_1, r_2, r_3\}$ and G^σ be the suspension of a σ-internally 3-connected plane graph G. A* Schnyder wood *of G^σ is an orientation and coloring of the edges of G^σ (including the half-edges) with the colors 1,2,3 (red, green, blue) such that*

1. *Every edge e is oriented in one direction (we say e is* unidirected*) or in two opposite directions (we say e is* bidirected*). Every direction of an edge is colored with one of the three colors 1,2,3 (we say an edge is i-*colored *if one of its directions has color i) such that the two colors i and j of every bidirected edge are distinct (we call such an edge i-j-*colored*). Throughout the paper, we assume modular arithmetic on the colors 1,2,3.*
2. *For every color i, the half-edge at r_i is unidirected, outgoing and i-colored.*
3. *Every vertex v has exactly one outgoing edge of every color. The outgoing 1-, 2-, 3-colored edges e_1, e_2, e_3 of v occur in clockwise order around v. For every color i, every incoming i-colored edge of v is contained in the clockwise sector around v from e_{i+1} to e_{i-1} (Figure 1i).*
4. *No inner face boundary contains a directed cycle in one color.*



(i) Properties of Schnyder woods. Condition 13 at a vertex.

(ii) The completion of G obtained by superimposing G^σ and its suspended dual G^{σ^*} (the latter depicted with dotted edges). The primal Schnyder wood is not the minimal element of the lattice of Schnyder woods of G, as this completion contains a clockwise directed cycle (marked in yellow).

Figure 1: Illustration for the definition of (dual) Schnyder woods.

For a Schnyder wood and color i, let T_i be the directed graph that is induced by the directed edges of color i. The following result justifies the name of Schnyder woods.

Lemma 1 ([13, 14])**.** *For every color i of a Schnyder wood of G^σ, T_i is a directed spanning tree of G in which all edges are oriented towards the root r_i.*

Lemma 2 (Felsner [15])**.** *Let T_i^{-1} be obtained from T_i by reversing the orientation of all its edges. For every $i \in \{1, \ldots, 3\}$, $T_i^{-1} \cup T_{i+1}^{-1} \cup T_{i+2}$ is acyclic.*

2.2. Dual Schnyder Woods

Let G be a σ-internally 3-connected plane graph. Any Schnyder wood of G^σ induces a Schnyder wood of a slightly modified planar dual of G^σ in the following way [16, 17] (see [18, p. 30] for an earlier variant of this result given without proof). As common for plane duality, we will use the plane dual operator * to switch between primal and dual objects (also on sets of objects).

Extend the three half-edges of G^σ to non-crossing infinite rays and consider the planar dual of this plane graph. Since the infinite rays partition the outer face f of G into three parts, this dual contains a triangle with vertices b_1, b_2 and b_3 instead of the outer face vertex f^* such that b_i^* is not incident to r_i for every i (Figure 1ii). Let the *suspended dual* G^{σ^*} of G^σ be the graph obtained from this dual by adding at each vertex of $\{b_1, b_2, b_3\}$ a half-edge pointing into the outer face.

Consider the superposition of G^σ and its suspended dual G^{σ^*} such that exactly the primal dual pairs of edges cross (here, for every $1 \leq i \leq 3$, the half-edge at r_i crosses the dual edge $b_{i-1}b_{i+1}$).

Definition 2. *For any Schnyder wood S of G^σ, define the orientation and coloring S^* of the suspended dual G^{σ^*} as follows (Figure 1ii):*

1. *For every unidirected $(i-1)$-colored edge or half-edge e of G^σ, color e^* with the two colors i and $i+1$ such that e points to the right of the i-colored direction.*
2. *Vice versa, for every i-$(i+1)$-colored edge e of G^σ, $(i-1)$-color e^* unidirected such that e^* points to the right of the i-colored direction.*
3. *For every color i, make the half-edge at b_i unidirected, outgoing and i-colored.*

The following lemma states that S^* is indeed a Schnyder wood of the suspended dual. The vertices b_1, b_2 and b_3 are called the *roots* of S^*.

Lemma 3 ([19][17, Prop. 3])**.** *For every Schnyder wood S of G^σ, S^* is a Schnyder wood of G^{σ^*}.*

Since $S^{*^*} = S$, Lemma 3 gives a bijection between the Schnyder woods of G^σ and the ones of G^{σ^*}. Let the *completion* $\widetilde{G}$ of G be the plane graph obtained from the superposition of G^σ and G^{σ^*} by subdividing each pair of crossing (half-)edges with a new vertex, which we call a *crossing vertex* (Figure 1ii).

Any Schnyder wood S of G^σ implies the following natural orientation and coloring $\widetilde{G}_S$ of its completion $\widetilde{G}$: For any edge $vw \in E(G^\sigma) \cup E(G^{\sigma^*})$, let z be the crossing vertex of G^σ that subdivides vw and consider the coloring of vw in either S or S^*. If vw is outgoing of v and i-colored, we direct $vz \in E(\widetilde{G})$ toward z and i-color it; analogously, if vw is outgoing of w and j-colored, we direct $wz \in E(\widetilde{G})$ toward z and j-color it. In the case that vw is unidirected, say w.l.o.g. incoming at v and i-colored, we direct $zv \in E(\widetilde{G})$ toward v and i-color it. The three half-edges of G^{σ^*} inherit the orientation and coloring of S^* for $\widetilde{G}_S$. By Definition 2, the construction of $\widetilde{G}_S$ implies immediately the following corollary.

Corollary 1. *Every crossing vertex of $\widetilde{G}_S$ has one outgoing edge and three incoming edges and the latter are colored 1, 2 and 3 in counterclockwise direction.*

Using results on orientations with prescribed outdegrees on the respective completions, Felsner and Mendez [20, 14] showed that the set of Schnyder woods of a planar suspension G^σ forms a distributive lattice. The order relation of this lattice relates a Schnyder wood of G^σ to a second Schnyder wood if the former can be obtained from the latter by reversing the orientation of a directed clockwise cycle in the completion. This gives the following lemma.

Lemma 4 ([20, 14])**.** *For the minimal element S of the lattice of all Schnyder woods of G^σ, $\widetilde{G}_S$ contains no clockwise directed cycle.*

We call the minimal element of the lattice of all Schnyder woods of G^σ the *minimal Schnyder wood* of G^σ.

2.3. Ordered path partitions

We denote paths as tuples of vertices such that consecutive vertices in the tuple are adjacent in the path. If a path P consists of only one vertex x, we might also write $P = x$. The concatenation of two paths P_1 and P_2 we denote by P_1P_2.

Definition 3. *For any $j \in \{1, 2, 3\}$ and any $\{r_1, r_2, r_3\}$-internally 3-connected plane graph G, an* ordered path partition $\mathcal{P} = (P_0, \ldots, P_s)$ *of G* with base-pair (r_j, r_{j+1}) *is a tuple of induced paths such that their vertex sets partition $V(G)$ and the following holds for every $i \in \{0, \ldots, s-1\}$, where $V_i := \bigcup_{q=0}^{i} V(P_q)$ and the contour C_i is the clockwise walk from r_{j+1} to r_j on the outer face of $G[V_i]$.*

1. *P_0 is the clockwise path from r_j to r_{j+1} on the outer face boundary of G, and $P_s = r_{j+2}$.*
2. *Every vertex in P_i has a neighbor in $V(G) \setminus V_i$.*
3. *C_i is a path.*
4. *Every vertex in C_i has at most one neighbor in P_{i+1}.*

Remark 1. *Our definition of an ordered path partition $\mathcal{P} = (P_0, \ldots, P_s)$ is essentially the definition of Badent et al. [21], in which the paths P_i need to be induced (this is not explicitly stated in [21], but used in the proof of their Theorem 5).*

By Definition 31 and 32, G contains for every i and every vertex $v \in P_i$ a path from v to r_{j+2} that intersects V_i only in v. Since G is plane, we conclude the following.

Lemma 5. *Every path P_i of an ordered path partition is embedded into the outer face of $G[V_{i-1}]$ for every $1 \le i \le s$.*

2.3.1. Compatible Ordered Path Partitions

We describe a connection between Schnyder woods and ordered path partitions that was first given by Badent et al. [21, Theorem 5] and then revisited by Alam et al. [22, Lemma 1].

Definition 4. *Let $j \in \{1, 2, 3\}$ and S be any Schnyder wood of the suspension G^σ of G. As proven in [22, arXiv version, Section 2.2], the inclusion-wise maximal j-$(j+1)$-colored paths of S then form an ordered path partition of G with base pair (r_j, r_{j+1}), whose order is a linear extension of the partial order given by reachability in the acyclic graph $T_j^{-1} \cup T_{j+1}^{-1} \cup T_{j+2}$; we call this special ordered path partition* compatible *with S and denote it by $\mathcal{P}^{j,j+1}$.*

For example, for the Schnyder wood given in Figure 1ii, $\mathcal{P}^{2,3}$ consists of the six maximal 2-3-colored paths, of which four are single vertices. We denote each path $P_i \in \mathcal{P}^{j,j+1}$ by $P_i := (v_1^i, \ldots, v_k^i)$ such that $v_1^i v_2^i$ is outgoing j-colored at v_1^i.

Let C_i be as in Definition 3. By Definition 33 and Lemma 5, every path $P_i = (v_1^i, \ldots, v_k^i)$ of an ordered path partition satisfying $i \in \{1, \ldots, s\}$ has a neighbor $v_0^i \in C_{i-1}$ that is closest to r_{j+1} and a different neighbor $v_{k+1}^i \in C_{i-1}$ that is closest to r_j. We call v_0^i the *left neighbor* of

P_i, v^i_{k+1} the *right neighbor* of P_i and $P^e_i := v^i_0 P_i v^i_{k+1}$ the *extension* of P_i; we omit superscripts if these are clear from the context. For $0 < i \leq s$, let the path P_i *cover* an edge e or a vertex x if e or x is contained in C_{i-1}, but not in C_i, respectively.

3. Bound on vertices of degree at least four

For the remainder of this section let G be a $\{r_1, r_2, r_3\}$-internally 3-connected plane graph of order n with a dual graph of order n' and a minimal Schnyder wood S. Define $B_{j,j+1}$ ($B'_{j,j+1}$) to be the number of singletons in $\mathcal{P}^{j,j+1}$ of the primal (dual) graph and $A_{j,j+1}$ ($A'_{j,j+1}$) to be the number of paths in $\mathcal{P}^{j,j+1}$ of the primal (dual) graph. We show that we can find a tree and co-tree with maximum degree at most four such that the number of vertices with degree four in the tree is at most $\min\{B'_{2,3}, n'+2-A'_{3,1}, n'+2-A'_{1,2}\}-1$ and the number of vertices with degree four in the co-tree is at most $\min\{B_{2,3}, n-A_{3,1}, n-A_{1,2}\}-1$.

We start with two lemmas on the structure of Schnyder woods. Then, we define our candidate graphs H and H' that have the same structural properties. We show that they both have maximum degree at most 3 and that for every edge of G either the edge itself is in H or its dual is in H'. We observe that for a cycle C in H the path with the highest index in $\mathcal{P}^{2,3}$ that contains a vertex of C needs to be a singleton. This is the key observation that leads to the upper bound on the number of degree-4-vertices. Then, we eventually prove the main theorem. The proof of the main theorem uses similar tools as presented in [11]. However, our candidate graph H which is different from the one used in [11] and the aforementioned observation additionally yield the upper bound on the number of vertices of degree 4.

Lemma 6 ([11])**.** *Let $\mathcal{P}^{2,3} = (P_0, \ldots, P_s)$ be the ordered path partition that is compatible with S. Let $P_i := (v_1, \ldots, v_k) \neq P_0$ be a path of $\mathcal{P}^{2,3}$ and v_0 and v_{k+1} be its left and right neighbor. Then, $v_0v_1, v_kv_{k+1} \in V(G)$, every edge $v_lw \notin \{v_0v_1, v_kv_{k+1}\}$ with $v_l \in P_i$ and $w \in V_{i-1}$ is unidirected, 1-colored and incoming at v_k and w is contained in the path of C_{i-1} between v_0 and v_{k+1} excluding both.*

Lemma 7 (Di Battista et al. [16])**.** *The boundary of every internal face of G can be partitioned into six paths $P_{1,3}$, $p_{2,3}$, $P_{2,1}$, $p_{3,1}$, $P_{3,2}$ and $p_{1,2}$ which appear in that clockwise order. For those paths the following holds (Figure 2).*

1. *$P_{i,j}$ consists of one edge which is either unidirected i-colored, unidirected j-colored or i-j-colored. Color i is directed in clockwise direction and color j in counterclockwise direction around f.*
2. *$p_{i,j}$ consists of a possibly empty sequence of i-j-colored edges such that color i is directed clockwise around f.*

Definition 5. *Let f be an internal face. Define $P_f = (x_1, \ldots, x_l)$ to be the path consisting of the edges on the boundary of f that are 2-3-colored or unidirected 3-colored such that color 3 is directed counterclockwise around f. By Lemma 7, P_f is indeed a path. It consists of $p_{2,3}$ and possibly $P_{1,3}$ (Figure 2). Let P_f be such that color 3 is directed from x_1 to x_l. For a vertex s let $cf(s)$ be the neighbor such that $cf(s)s$ is the clockwise first incoming 1-colored edge at s. Define H to be the subgraph of G with vertex set $V(G)$ and the edge set given by*

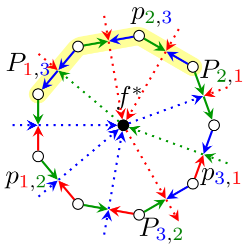


Figure 2: Illustration for Lemma 7. A face f, the paths on its boundary and the dual edges incident to f^*. P_f is marked in yellow.

1. *the 3-1-colored edges,*
2. *for every vertex s the edge $cf(s)s$,*
3. *for every internal face f all edges of $P_f = (x_1, \ldots, x_l)$ except for $x_{l-1}x_l$,*
4. *the 2-3-colored edges on the outer face.*

Observe that the edges added by Condition 3 are 2-3-colored. Define H' the same way for $G^{\sigma *}$. See Figure 3 for illustration.

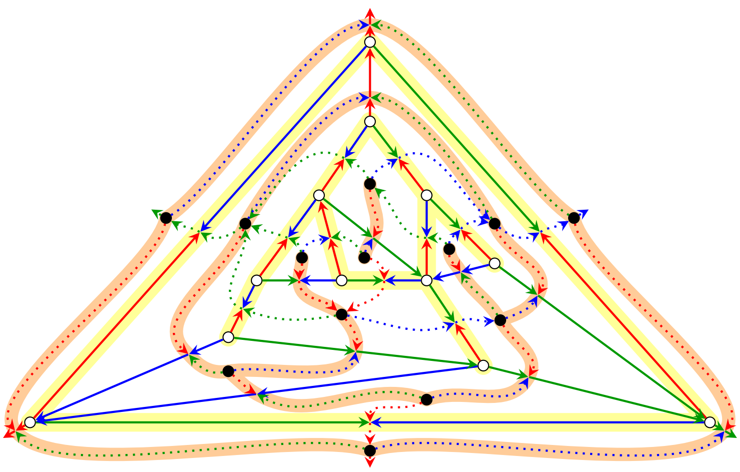


Figure 3: Illustration for the definition of H (depicted in yellow) and H' (depicted in orange).

Observe that for $P_f = (x_1, \ldots, x_l)$ we have $cf(f^*)f^* = (x_{l-1}x_l)^*$ (Figure 2). By Condition 3, $x_{l-1}x_l \notin E(H)$ and, by Condition 2, $(x_{l-1}x_l)^* \in E(H')$. Similar observations and a complete case distinction that covers all possible orientations and colorings of an edge of G then yield the following lemma.

Lemma 8. *For an edge $e \in E(G)$ we have that $e \in E(H)$ if and only if $e^* \notin E(H')$. And thus, $H' = \neg H^* \cup \{b_1b_2, b_2b_3, b_3b_1\}$.*

Lemma 9. *H and H' both have maximum degree at most 3.*

Proof. We show that H has maximum degree at most 3. The arguments work similarly for H'. Consider a vertex $v \in V(H)$.

Assume that v is incident to a 2-3-colored edge $e \in E(H)$ that is incoming 3-colored at v. Then, either Definition 53 or 4 applies to e. We give a short argument that in both cases there is no edge in the clockwise sector around v between e and the outgoing 3-colored edge. If Definition 54 applies to e, then v is on the clockwise path from r_2 to r_3 on the boundary of the outer face, and the claim obviously holds.

Otherwise, Definition 53 applies to e. Assume, for the sake of contradiction, that there is an edge in the clockwise interval around v between e and the outgoing 3-colored edge. Let e' be the clockwise first such edge. By Definition 13, e' is unidirected 1-colored and incoming at v. e and e' are on a common face f. The path $P_f = (x_1, \ldots, x_l)$ contains e. Since e' is not outgoing 3-colored at v, we obtain that $x_{l-1}x_l = e$ and thus, $e \notin E(H)$, a contradiction.

By Definition 13, the unidirected incoming 1-colored edges occur in the clockwise sector around v between e and the outgoing 3-colored edge. Thus, there is no unidirected incoming 1-colored edge at v. Hence, by Definition 5, the outgoing 1-colored edge and the outgoing 3-colored edge at v are the only additional edges incident to v that might be in $E(H)$. This yields that $\deg_H(v) \leq 3$.

So assume that v is not incident to a 2-3-colored edge $e \in E(H)$ that is incoming 3-colored at v. Then, by Definition 5, there are at most three edges incident to v that might be in $E(H)$, namely $cf(v)v$, the outgoing 3-colored and the outgoing 1-colored edge. Thus, we have $\deg_H(v) \leq 3$. ☐

Definition 6. *Let C be a cycle in H. Let $\mathcal{P}^{2,3} = (P_0, \ldots, P_s)$ be the compatible ordered path partition of S. Let P be the path of maximal length in C such that $P \subseteq P_M$ with $M := \max\{i \mid P_i \cap V(C) \neq \emptyset\}$. We call P the* index maximal subpath *of C. Denote by $\mathcal{P}_{max}$ the set of all index maximal subpaths.*

By Lemma 6, every path $P_i = (v_1, \ldots, v_k) \in \mathcal{P}^{2,3}$ is connected to V_{i-1} only by v_0v_1 and edges incident to v_k. Also, Definition 53 yields that if $k \geq 2$, then either v_0v_1 or v_1v_2 is not in H. Hence, we obtain the following lemma.

Lemma 10. *Let $P_i = (v_1, \ldots, v_k) \in \mathcal{P}^{2,3}$ be a path containing an index maximal subpath P of a cycle C in H. Then, P_i is a singleton, the edge from P_i to its left neighbor v_0 is 3-1-colored and in C and the other edge in C incident to v_1 is $cf(v_1)v_1$. The same holds for H'.*

Definition 7. *A singleton in $\mathcal{P}^{2,3}$ is a* 1-2-singleton *(*2-singleton*) if its outgoing 2-colored edge is 1-2-colored (unidirected) and its outgoing 3-colored edge is 3-1-colored.*

Observe that, by Lemma 10, index maximal subpaths are either 1-2-singletons or 2-singletons. And, for a 1-2-singleton, s the outgoing 2-colored edge and $cf(s)s$ coincide.

Theorem 1. *Let G be a $\{r_1, r_2, r_3\}$-internally 3-connected plane graph of order n with a dual graph of order n' and a minimal Schnyder wood S of G^σ. There is a 4-tree T in G such that $\neg T^*$ is a 4-tree. Also, the number of degree-4-vertices in T is at most $\min\{B'_{2,3}, n' + 2 - A'_{3,1}, n' + 2 - A'_{1,2}\} - 1$ and the number of degree-4-vertices in $\neg T^*$ is at most $\min\{B_{2,3}, n - A_{3,1}, n - A_{1,2}\} - 1$.*

Sketch of proof. Let H and H' be as defined in Definition 5. Recall that, by Lemma 8, $e \in E(H)$ if and only if $e^* \notin E(H')$, and thus $H' = \neg H^* \cup \{b_1b_2, b_2b_3, b_3b_1\}$. As b_1b_2, b_2b_3 and b_3b_1 are not in G^*, they do not affect our desired trees. By Lemma 9, H and H' both have maximum degree at most 3. Observe that H and H' might both have cycles and are not necessarily connected (Figure 3).

We will therefore iteratively identify edges of cycles of H such that $\neg H^* \cup \{b_1b_2, b_2b_3, b_3b_1\}$ still has maximum degree at most four when those edges are deleted in H. In order to do this, we iteratively define the set of edges $D \subseteq E(H)$ that is deleted from H. Then, we use the exact same arguments in order to define the set of edges D' that is deleted from H'. We start with $D = D' = \emptyset$.

Let $\mathcal{P}^{2,3} = (P_0, \dots, P_s)$ be the compatible ordered path partition formed by the maximal 2-3-colored paths. We will consider paths that are the first (i.e. index minimal) path covering an index maximal subpath. For a path $P \in \mathcal{P}_{max} \setminus \{P_s\}$ refer to P_L with $L = \min\{i \mid P_i$ covers an edge of the extension of $P\}$ as the *minimal-covering path* of P. Denote by $\mathcal{P}_{cover}$ the set of the minimal-covering paths of the paths of $\mathcal{P}_{max} \setminus \{P_s\}$.

Observe that $P_s = r_1$ is the index maximal subpath of the outer face boundary and a 1-2-singleton. There is no path in $\mathcal{P}^{2,3}$ that covers an edge of P_s. Hence, in order to destroy the outer face cycle, we add the outgoing 3-colored edge of r_1 to D.

Next, we process the paths of $\mathcal{P}_{cover}$ in reverse order of $\mathcal{P}^{2,3}$, i.e., from highest to lowest index. Let $P_c = (v_1, \dots, v_k) \in \mathcal{P}_{cover}$, $c \in \{1, \dots, s\}$ be the path under consideration. Let $s_1, \dots, s_l$ be the index maximal paths for which P_c is the minimal-covering path, ordered clockwise around the outer face of $G[V_{c-1}]$ (Figure 4). Let $f_1, \dots, f_a$ be the faces incident to v_k in counterclockwise order from the outgoing 3-colored edge to the outgoing 2-colored edge. We say that $f_1, \dots, f_a$ are *below* P_c and *above* $s_1, \dots, s_l$. Observe that since S is minimal, if $v_{k+1} = s_l$, then v_kv_{k+1} is 1-2-colored [11, proof of Theorem 14].

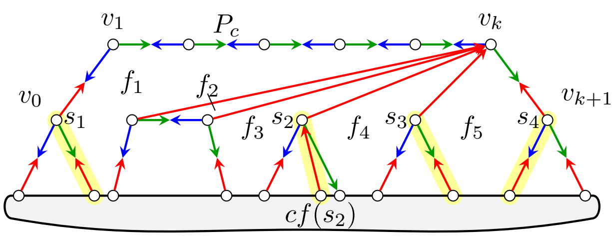


Figure 4: Illustration for some of the definitions used in the proof of Theorem 1. The situation if P_c is a path. The edges that we add to D are marked in yellow. In the depicted situation, v_1v_2 is not in H.

Remember that, by Lemma 10, $cf(s)s$ and the outgoing 3-colored edge are the only edges in H that join s with vertices of V_{i-1}.

Now, we select for each of the singletons $s \in \{s_1, \dots, s_l\}$ either the outgoing 3-colored edge or $cf(s)s$ and add it to D. Thus, after having processed every path in $\mathcal{P}_{cover}$, a cycle in H does not exist in $H - D$ anymore. We aim for selecting those edges that have the smallest possible impact on the maximum degree of the dual graph. Hence, for a 2-singleton s we always choose the edge $cf(s)s$. Deleting this specific edge does not increase the degree of any face below P_c (Figure 4). In detail we distinguish the following three cases.

Augmentation procedure of D for the path P_c:

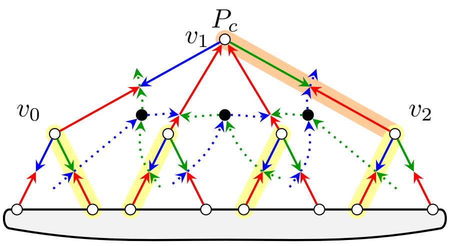
Case 1: $P_c = (v_1, \dots, v_k)$ is not in $\mathcal{P}_{max}$. For every singleton $s \in \{s_1, \dots, s_l\} \setminus \{v_{k+1}\}$, we add $cf(s)s$ to D. If $v_{k+1} = s_l$ is a 2-singleton, we add $cf(s_l)s_l$ to D. Otherwise, we add its outgoing 3-colored edge to D (Figure 4).

Case 2: $P_c = v_1$ is an index maximal subpath and a 1-2-singleton. Then, we already have either $v_0v_1 \in D$ or $v_1v_2 \in D$.

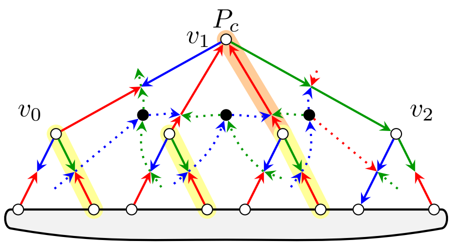
Case 2.1: $v_0v_1 \in D$. We proceed as in Case 1.

Case 2.2: $v_1v_2 \in D$. For every singleton $s \in \{s_1, \dots, s_l\}$ that is a 2-singleton, we add $cf(s)s$ to D. For every singleton $s \in \{s_1, \dots, s_l\} \setminus \{v_0\}$ that is a 1-2-singleton, we add the outgoing 3-colored edge to D. If $v_0 = s_1$ is a 1-2-singleton, then add its outgoing 2-colored edge to D (Figure 5i).

Case 3: $P_c = v_1$ is an index maximal subpath and a 2-singleton. Then, we already added $cf(v_1)v_1$ to D. For every singleton $s \in \{s_1, \dots, s_l\}$, we add $cf(s)s$ to D. Observe that v_1v_2 is unidirected 2-colored and hence $v_2 \neq s_l$ (Figure 5ii).



(i) The situation in Case 2.2.



(ii) The situation in Case 3.

Figure 5: Illustration for the augmentation procedure of D in the proof of Theorem 1. Edges that are marked in orange are in D before we consider P_c. The edges that we then add to D are marked in yellow.

As in [11, proof of Theorem 14] we observe that after having processed P_c no more edges incident to $f_1, \dots, f_a$ are added to D. By Definition 32, the boundary of every f_j with $j \in \{1, \dots, a\}$ contains at most two edges that are in the union of the extensions of singletons in $\{s_1, \dots, s_l\}$. In Case 1 and 2, for every $j \in \{2, \dots, a-1\}$, we add at most one edge of the boundary of f_j to D. This implies that $\deg_{\neg H^*+D^*}(f_j^*) \leq 4$ for every $j \in \{2, \dots, a-1\}$ (Figure 4 and 5i). The same holds for every $j \in \{1, \dots, a-2\}$ in Case 3 (Figure 5ii). Similar arguments as in [11, proof of Theorem 14] show that the degree of the remaining vertices of $f_1^*, \dots, f_a^*$ does not exceed 4 in $\neg H^* + D^*$.

Also, those arguments yield that every degree-4-vertex f^* of $\neg H^* + D^*$ such that f is below a path of $\mathcal{P}_{cover}$ has at least one 1-2-singleton x on its boundary such that f^* is above x and either the outgoing 2-colored edge at x or the outgoing 3-colored edge at x is on the boundary of f and in D (Figure 4 and 5). In the following, we show that the vertices of $\neg H^* + D^*$ that are not below a path of $\mathcal{P}_{cover}$ have maximum degree at most 3. Hence, the assignment of degree-4-vertices of $\neg H^* + D^*$ to 1-2-singletons implied by the above relation is injective.

Now, we show that a vertex f'^* of $\neg H^* + D^*$ such that f' is not below a path of $\mathcal{P}_{cover}$ has degree at most 3 in $\neg H^* + D^*$. If f'^* is not incident to an edge of D^*, then we have that $\deg_{\neg H^*+D^*}(f'^*) = \deg_{\neg H^*}(f'^*) \leq 3$. If f'^* is incident to an edge of D^*, then f'^* is below a path P of $\mathcal{P}_{max}$. Such a path is either a 1-2-singleton or a 2-singleton. Hence, f'^* does not have unidirected incoming 1-colored edges. We have that $P = u_1$ for a vertex $u_1 \in V(G)$. Assume that P is a 1-2-singleton. Then, either $u_0u_1 \in D$ or $u_1u_2 \in D$. And thus, f'^* is either incident to $(u_0u_1)^*$ or $(u_1u_2)^*$. If f'^* is incident to $(u_0u_1)^*$ ($(u_1u_2)^*$), then the only edges incident to f'^* that might be in $\neg H^*$ are its outgoing 2-colored (3-colored) edge and its outgoing 1-colored edge, i.e., $\deg_{\neg H^*}(f'^*) \leq 2$ and thus $\deg_{\neg H^*+D^*}(f'^*) \leq 3$. So assume that P is a 2-singleton. Then, f'^* is incident to $(cf(u_1)u_1)^*$. This dual edge is 2-3-colored. As f'^* does not have unidirected incoming 1-colored edges, the edges incident to f'^* that are potentially in $\neg H^* + D^*$ are its outgoing 2-colored, its outgoing 3-colored and its outgoing 1-colored edge (Figure 5ii, but ignore the edges marked in yellow). We obtain that $\deg_{\neg H^*+D^*}(f'^*) \leq 3$.

This yields that each degree-4-vertex f^* in $\neg H^* + D^*$ has at least one 1-2-singleton x of $\mathcal{P}^{2,3}$ of G on its boundary such that f^* is above x and either the outgoing 2-colored edge at x or the outgoing 3-colored edge at x is on the boundary of f and in D. We assign each degree-4-vertex to such a 1-2-singleton. Since we never add both the outgoing 2-colored and the outgoing 3-colored edge of a 1-2-singleton to D, this assignment is injective. Also, we know that r_1 is a 1-2-singleton but as there is no path in $\mathcal{P}_{cover}$ covering r_1, no degree-4-vertex is assigned to r_1. Thus, we obtain that the number of degree-4-vertices in $\neg H^* + D^*$ is at most the number of 1-2-singletons minus one. Every 1-2-singleton is incident to a 3-1-colored and a 1-2-colored edge that are both incoming 1-colored. The number of 3-1-colored and a 1-2-colored edges is at most $n - A_{3,1}$ and $n - A_{1,2}$, respectively. Hence, we obtain that the number of degree-4-vertices in $\neg H^* + D^*$ is at most $\min\{B_{2,3}, n - A_{3,1}, n - A_{1,2}\} - 1$.

So far we showed that $H - D$ is acyclic, $\neg H^* + D^*$ has maximum degree at most 4 and that the claimed upper bound on the number of degree-4-vertices of $H - D$ holds. We now apply the same arguments that we used for H to $H' = \neg H^* \cup \{b_1b_2, b_2b_3, b_3b_1\}$ (equality by Lemma 8) and obtain D'. Hence, we have that $H' - D'$ is acyclic and $H + D'^* \setminus \{b_1b_2, b_2b_3, b_3b_1\}^*$ has maximum degree at most 4. Since G^* and $G^{\sigma *}$ differ on the outer face we obtain the slightly different upper bound of $\min\{B'_{2,3}, n' + 2 - A'_{3,1}, n' + 2 - A'_{1,2}\} - 1$ for the number of degree-4-vertices of $H + D'^* \setminus \{b_1b_2, b_2b_3, b_3b_1\}^*$.

The edges b_1b_2, b_2b_3 and b_3b_1 are not in G^* and there is only one edge on the boundary of the outer face of G that is also in D. We may thus ignore b_1b_2, b_2b_3 and b_3b_1 and freely switch from $\neg H^* \cup \{b_1b_2, b_2b_3, b_3b_1\}$ to $\neg H^*$. Hence, we also remove any of the edges b_1b_2, b_2b_3, b_3b_1 from D'. Similar arguments as in [11, proof of Theorem 14] yield that $H - D + D'^*$ and $\neg H^* - D' + D^*$ are both acyclic. As $\neg H^* - D' + D^* = \neg(H - D + D'^*)^*$, this yields that they are also both connected. Hence, they are the desired tree and its co-tree. □

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