Toward Grünbaum's Conjecture

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Abstract

Given a spanning tree T of a planar graph G, the *co-tree* of T is the spanning tree of the dual graph G^* with edge set $(E(G) - E(T))^*$. Grünbaum conjectured in 1970 that every planar 3-connected graph G contains a spanning tree T such that both T and its co-tree have maximum degree at most 3.

While Grünbaum's conjecture remains open, Biedl proved that there is a spanning tree T such that T and its co-tree have maximum degree at most 5. By using new structural insights into Schnyder woods, we prove that there is a spanning tree T such that T and its co-tree have maximum degree at most 4. This tree can be computed in linear time.

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1 Introduction

Let a *k-tree* be a spanning tree whose maximum degree is at most k. In 1966, Barnette proved the fundamental theorem that every planar 3-connected graph contains a 3-tree [3]. Both assumptions in this theorem are essential in the sense that the statement fails for arbitrary non-planar graphs (as the arbitrarily high degree in any spanning tree of the complete bipartite graphs $K_{3,n-3}$ show) as well as for graphs that are not 3-connected (as the planar graphs $K_{2,n-2}$ show).

Since then, Barnette's theorem has been extended and generalized in several directions. First, one may try to relax the 3-connectedness assumption: Indeed, Barnette's original proof holds for the slightly more general class of *circuit graphs*[1], and may also be extended to arbitrary planar graphs G in form of a local version that guarantees for every 3-connected[2] vertex set X of G a (not necessarily spanning) tree of G that has maximum degree at most 3 and contains X [6]. Alternatively, one may relax the planarity assumption. Ota and Ozeki [22] proved that for every $k \geq 3$, every 3-connected graph with no $K_{3,k}$-minor contains a $(k-1)$-tree if k is even and a k-tree if k is odd. Further sufficient conditions for the existence of k-trees may be found in the survey [23].

Second, one may see spanning trees as 1-connected spanning subgraphs and generalize these to k-connected spanning subgraphs for any $k > 1$. In this direction, Barnette [4] proved that every planar 3-connected planar graph contains a 2-connected spanning subgraph whose maximum degree is at most 15, and Gao [17] improved this result subsequently to the tight bound of maximum degree at most 6. Interestingly, Gao showed that his result holds as well for the 3-connected graphs that are embeddable on the projective plane, the torus or the Klein bottle.

[1] that is, planar internally 3-connected graphs with a designated outer face
[2] $X \subseteq V(G)$ such that G contains three internally vertex-disjoint paths between every two vertices of X

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Third, one may try to strengthen the 3-tree in question. A recent alternative proof of Barnette's theorem based on canonical orderings by Biedl [5, Corollary 1] (which was also mentioned by Chrobak and Kant) reveals that further degree constraints may be imposed on the 3-tree for prescribed vertices (for example, two vertices of a common face may be forced to be leaves of the tree). To strengthen this further, Barnette's theorem can be seen as a side-result of a structure obtained in Hamiltonicity studies from generalizing the theory of Tutte paths and Tutte cycles: Gao and Richter [18] proved that every planar 3-connected graph contains a 2-*walk*, which is a walk that visits every vertex exactly once or twice. By going along such 2-walks and omitting the last edge whenever a vertex is revisited, these 2-walks imply the existence of 3-trees. Here, planar 3-connected graphs may again be replaced with circuit graphs, and all results have been successfully lifted to higher surfaces. Even more, the surfaces on which every embedded 3-connected graph contains a 2-walk have been classified [7].

Perhaps one of the most severe strengthenings of the 3-tree in question is a long-standing and to the best of our knowledge still open conjecture made by Grünbaum in 1970. Since the planar dual $G^* = (V^*, E^*)$ of every (simple) planar 3-connected graph G is again planar and 3-connected, G^* contains a 3-tree as well. By the well-known cut-cycle duality, any spanning tree T of G implies that also $\neg T^* := (V^*, (E(G) - E(T))^*)$ is a spanning tree of G^*; we call $\neg T^*$ the *co-tree* of T. Taking the best of these two worlds, Grünbaum made the following conjecture.

▶ **Conjecture** (Grünbaum [19, p. 1148], 1970)**.** *Every planar 3-connected graph G contains a 3-tree T whose co-tree $\neg T^*$ is also a 3-tree.*

While Grünbaum's conjecture is to the best of our knowledge still unsolved, progress has been made by Biedl [5], who proved the existence of a 5-tree, whose co-tree is a 5-tree. We prove the existence of a 4-tree, whose co-tree is a 4-tree. Our methods exploit insights into the structure of Schnyder woods. We discuss Schnyder woods, their lattice structure and ordered path partitions in Section 2, our main result in Section 3 and computational aspects of this main result in Section 5.

2 Schnyder Woods and Ordered Path Partitions

We only consider simple undirected graphs. A graph is *plane* if it is planar and embedded into the Euclidean plane without intersecting edges. The *neighborhood of a vertex set A* is the union of the neighborhoods of vertices in A. Although parts of this paper use orientation on edges, we will always let vw denote the undirected edge $\{v, w\}$.

2.1 Schnyder Woods

Let $\sigma := \{r_1, r_2, r_3\}$ be a set of three vertices of the outer face boundary of a plane graph G in clockwise order (but not necessarily consecutive). We call r_1, r_2 and r_3 *roots*. The *suspension G^σ* of G is the graph obtained from G by adding at each root of σ a half-edge pointing into the outer face. With a little abuse of notation, we define a *half-edge* as an arc that has a startvertex but no endvertex. A plane graph G is *σ-internally 3-connected* if the graph obtained from the suspension G^σ of G by making the three half-edges incident to a common new vertex inside the outer face is 3-connected. Note that the class of σ-internally 3-connected plane graphs properly contains all 3-connected plane graphs.

▶ **Definition 1** (Felsner [11]). *Let $\sigma = \{r_1, r_2, r_3\}$ and G^σ be the suspension of a σ-internally 3-connected plane graph G. A* Schnyder wood *of G^σ is an orientation and coloring of the edges of G^σ (including the half-edges) with the colors 1,2,3 (red, green, blue) such that*

(a) *Every edge e is oriented in one direction (we say e is* unidirected*) or in two opposite directions (we say e is* bidirected*). Every direction of an edge is colored with one of the three colors 1,2,3 (we say an edge is i-*colored *if one of its directions has color i) such that the two colors i and j of every bidirected edge are distinct (we call such an edge i-j-*colored*). Similarly, a unidirected edge whose direction has color i is called i-*colored. *Throughout the paper, we assume modular arithmetic on the colors 1,2,3 in such a way that $i+1$ and $i-1$ for a color i are defined as $(i \mod 3)+1$ and $(i+1 \mod 3)+1$. For a vertex v, a uni- or bidirected edge is* incoming *(i-colored) in v if it has a direction (of color i) that is directed toward v, and* outgoing *(i-colored) of v if it has a direction (of color i) that is directed away from v.*

(b) *For every color i, the half-edge at r_i is unidirected, outgoing and i-colored.*

(c) *Every vertex v has exactly one outgoing edge of every color. The outgoing 1-, 2-, 3-colored edges e_1, e_2, e_3 of v occur in clockwise order around v. For every color i, every incoming i-colored edge of v is contained in the clockwise sector around v from e_{i+1} to e_{i-1} (see Figure 1).*

(d) *No inner face boundary contains a directed cycle (disregarding possible opposite edge directions) in one color.*

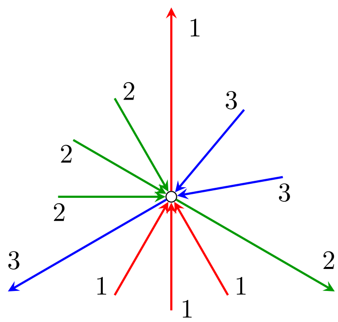


Figure 1 Properties of Schnyder woods. Condition 1c at a vertex.

For a Schnyder wood and color i, let T_i be the directed graph that is induced by the directed edges of color i. The following result justifies the name of Schnyder woods.

▶ **Lemma 2** ([12, 24]). *For every color i of a Schnyder wood of G^σ, T_i is a directed spanning tree of G in which all edges are oriented to the root r_i.*

For a directed graph H, we denote by H^{-1} the graph obtained from H by reversing the direction of all its edges.

▶ **Lemma 3** (Felsner [14]). *For every $i \in \{1, \ldots, 3\}$, $T_i^{-1} \cup T_{i+1}^{-1} \cup T_{i+2}$ is acyclic.*

2.2 Dual Schnyder Woods

Let G be a σ-internally 3-connected plane graph. Any Schnyder wood of G^σ induces a Schnyder wood of a slightly modified planar dual of G^σ in the following way [9, 13] (see [21, p. 30] for an earlier variant of this result given without proof). As common for plane duality, we will use the plane dual operator * to switch between primal and dual objects (also on sets of objects).

Extend the three half-edges of G^σ to non-crossing infinite rays and consider the planar dual of this plane graph. Since the infinite rays partition the outer face f of G into three parts, this dual contains a triangle with vertices b_1, b_2 and b_3 instead of the outer face vertex f^* such that b_i^* is not incident to r_i for every i (see Figure 2). Let the *suspended dual* G^{σ^*} of G^σ be the graph obtained from this dual by adding at each vertex of $\{b_1, b_2, b_3\}$ a half-edge pointing into the outer face.

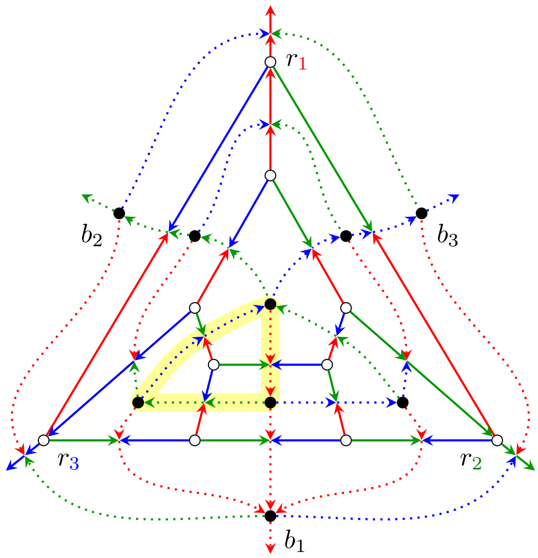


Figure 2 The completion of G obtained by superimposing G^σ and its suspended dual G^{σ^*} (the latter depicted with dotted edges). The primal Schnyder wood is not the minimal element of the lattice of Schnyder woods of G, as this completion contains a clockwise directed cycle (marked in yellow).

Consider the superposition of G^σ and its suspended dual G^{σ^*} such that exactly the primal dual pairs of edges cross (here, for every $1 \leq i \leq 3$, the half-edge at r_i crosses the dual edge $b_{i-1}b_{i+1}$).

▶ **Definition 4.** *For any Schnyder wood S of G^σ, define the orientation and coloring S^* of the suspended dual G^{σ^*} as follows (see Figure 2):*

(a) *For every unidirected $(i-1)$-colored edge or half-edge e of G^σ, color e^* with the two colors i and $i+1$ such that e points to the right of the i-colored direction.*

(b) *Vice versa, for every i-$(i+1)$-colored edge e of G^σ, $(i-1)$-color e^* unidirected such that e^* points to the right of the i-colored direction.*

(c) *For every color i, make the half-edge at b_i unidirected, outgoing and i-colored.*

The following lemma states that S^* is indeed a Schnyder wood of the suspended dual. The vertices b_1, b_2 and b_3 are called the *roots* of S^*.

▶ **Lemma 5** ([20], [13, Prop. 3])**.** *For every Schnyder wood S of G^σ, S^* is a Schnyder wood of G^{σ^*}.*

Since $S^{*^*} = S$, Lemma 5 gives a bijection between the Schnyder woods of G^σ and the ones of G^{σ^*}. Let the *completion* $\widetilde{G}$ of G be the plane graph obtained from the superposition of G^σ and G^{σ^*} by subdividing each pair of crossing (half-)edges with a new vertex, which we call a *crossing vertex* (see Figure 2). The completion has six half-edges pointing into its outer face.

Any Schyder wood S of G^σ implies the following natural orientation and coloring $\widetilde{G}_S$ of its completion $\widetilde{G}$: For any edge $vw \in E(G^\sigma) \cup E(G^{\sigma^*})$, let z be the crossing vertex of G^σ that subdivides vw and consider the coloring of vw in either S or S^*. If vw is outgoing of v and i-colored, we direct $vz \in E(\widetilde{G})$ toward z and i-color it; analogously, if vw is outgoing of w and j-colored, we direct $wz \in E(\widetilde{G})$ toward z and j-color it. In the case that vw is unidirected, say without loss of generality incoming at v and i-colored, we direct $zv \in E(\widetilde{G})$ toward v and i-color it. The three half-edges of G^{σ^*} inherit the orientation and coloring of S^* for $\widetilde{G}_S$. By Definition 4, the construction of $\widetilde{G}_S$ implies immediately the following corollary.

▶ **Corollary 6.** *Every crossing vertex of $\widetilde{G}_S$ has one outgoing edge and three incoming edges and the latter are colored 1, 2 and 3 in counterclockwise direction.*

Using results on orientations with prescribed outdegrees on the respective completions, Felsner and Mendez [8, 12] showed that the set of Schnyder woods of a planar suspension G^σ forms a distributive lattice. The order relation of this lattice relates a Schnyder wood of G^σ to a second Schnyder wood if the former can be obtained from the latter by reversing the orientation of a directed clockwise cycle in the completion. This gives the following lemma, of which the computational part is due to Fusy [15].

▶ **Lemma 7** ([8, 12, 15]). *For the minimal element S of the lattice of all Schnyder woods of G^σ, $\widetilde{G}_S$ contains no clockwise directed cycle. Also, S and $\widetilde{G}_S$ can be computed in linear time.*

We call the minimal element of the lattice of all Schnyder woods of G^σ the *minimal Schnyder wood* of G^σ.

2.3 Ordered Path Partitions

▶ **Definition 8.** *For any $j \in \{1, 2, 3\}$ and any $\{r_1, r_2, r_3\}$-internally 3-connected plane graph G, an* ordered path partition $\mathcal{P} = (P_0, \dots, P_s)$ *of G* with base-pair (r_j, r_{j+1}) *is an ordered partition of $V(G)$ into the vertex sets of induced paths such that the following holds for every $i \in \{0, \dots, s-1\}$, where $V_i := \bigcup_{q=0}^{i} P_q$ and the* contour C_i *is the clockwise walk from r_{j+1} to r_j on the outer face of $G[V_i]$.*

(a) *P_0 is the vertex set of the clockwise path from r_j to r_{j+1} on the outer face boundary of G, and $P_s = \{r_{j+2}\}$.*
(b) *Every vertex in P_i has a neighbor in $V(G) \setminus V_i$.*
(c) *C_i is a path.*
(d) *Every vertex in C_i has at most one neighbor in P_{i+1}.*

For the ease of notation we often refer to vertex sets of paths as paths.

▶ Remark 9. Our definition of an ordered path partition $\mathcal{P} = (P_0, \dots, P_s)$ is essentially the definition of Badent et al. [2], in which the vertex sets P_i have to induce paths (this is not explicitly stated in [2], but used in the proof of their Theorem 5). Because a part of the proof of Theorem 5 in [2] (correspondence of ordered path partitions and Schnyder woods) was incomplete, Alam et al. [1, Lemma 1] corrected the result, but unfortunately outsourced the corrected proof into the extended abstract [1, arXiv version, Section 2.2] only. In this correction however, Alam et al. [1] give an incomplete definition[3] of ordered path partitions that misses Condition b. This incompleteness does however not affect the proof of their Lemma 4 [1, arXiv version], as this only gives a correction of [2, Theorem 5] regarding the order of the paths. In this paper, we only use Lemma 4 of [1, arXiv version] which is identical to [1, Lemma 1].

[3] Confirmed by personal communication with the authors of [1].

By Definition 8a and 8b, G contains for every i and every vertex $v \in P_i$ a path from v to r_{j+2} that intersects V_i only in v. Since G is plane, we conclude the following.

▶ **Lemma 10.** *Every path P_i of an ordered path partition is embedded into the outer face of $G[V_{i-1}]$ for every $1 \leq i \leq s$.*

Compatible Ordered Path Partitions

We describe a connection between Schnyder woods and ordered path partitions that was first given by Badent et al. [2, Theorem 5] and then revisited by Alam et al. [1, Lemma 1].

▶ **Definition 11.** *Let $j \in \{1, 2, 3\}$ and S be any Schnyder wood of the suspension G^σ of G. As proven in [1, arXiv version, Section 2.2], the vertex sets of the inclusion-wise maximal j-$(j+1)$-colored paths of S then form an ordered path partition of G with base pair (r_j, r_{j+1}), whose order is a linear extension of the partial order given by reachability in the acyclic graph $T_j^{-1} \cup T_{j+1}^{-1} \cup T_{j+2}$; we call this special ordered path partition* compatible *with S and denote it by $\mathcal{P}^{j,j+1}$.*

For example, for the Schnyder wood given in Figure 2, $\mathcal{P}^{2,3}$ consists of the vertex sets of six maximal 2-3-colored paths, of which four are single vertices. We denote each path $P_i \in \mathcal{P}^{j,j+1}$ by $P_i := \{v_1^i, \ldots, v_k^i\}$ such that $v_1^i v_2^i$ is outgoing j-colored at v_1^i and, for every $l \in \{1, \ldots, k-1\}$, $v_l^i v_{l+1}^i$ is a j-$(j+1)$-colored edge.

Let C_i be as in Definition 8. By Definition 8c and Lemma 10, every path $P_i = \{v_1^i, \ldots, v_k^i\}$ of an ordered path partition satisfying $i \in \{1, \ldots, s\}$ has a neighbor $v_0^i \in C_{i-1}$ that is closest to r_{j+1} and a different neighbor $v_{k+1}^i \in C_{i-1}$ that is closest to r_j (see Figure 3). We call v_0^i the *left neighbor* of P_i, v_{k+1}^i the *right neighbor* of P_i and $P_i^e := \{v_0^i\} \cup P_i \cup \{v_{k+1}^i\}$ the *extension* of P_i; we omit superscripts if these are clear from the context. For $0 < i \leq s$, let the path P_i *cover* an edge e or a vertex x if e or x is contained in C_{i-1}, but not in C_i, respectively.

▶ **Lemma 12.** *Every path $P_i \neq P_0$ of a compatible ordered path partition $\mathcal{P}^{j,j+1}$ satisfies the following (see Figure 3):*

(a) *Every neighbor of P_i that is in V_{i-1} is contained in the path of C_{i-1} between v_0^i and v_{k+1}^i.*
(b) *$v_0^i v_1^i$ and $v_k^i v_{k+1}^i$ are edges of $G[V_i]$.*
(c) *$v_0^i v_1^i$ is $(j+1)$-colored outgoing at v_1^i and $v_k^i v_{k+1}^i$ is j-colored outgoing at v_k^i.*
(d) *Every edge $v_l^i x$ incident to P_i and V_{i-1} except for $v_0^i v_1^i$ and $v_k^i v_{k+1}^i$ is unidirected toward P_i, $(j+2)$-colored and satisfies $x \notin \{v_0^i, v_{k+1}^i\}$.*

Proof. The statement a follows directly from Lemma 10 and the definition of left and right neighbor of P_i.

Now, we prove statements b and c. According to Definition 11, the order of $\mathcal{P}^{j,j+1}$ on the vertex sets of paths is a linear extension of the partial order given by reachability in the acyclic graph $T_j^{-1} \cup T_{j+1}^{-1} \cup T_{j+2}$. This allows us to characterize the color of the edges that join P_i with vertices of V_{i-1} and $V - V_i$, respectively. Edges that join P_i with vertices of V_{i-1} are incoming $(j+2)$-colored, unidirected outgoing j-colored or unidirected outgoing $(j+1)$-colored at a vertex of P_i. Edges that join P_i with vertices of $V - V_i$ are outgoing $(j+2)$-colored, unidirected incoming j-colored or unidirected incoming $(j+1)$-colored at a vertex of P_i.

Recall that all edges of $G[P_i]$ are j-$(j+1)$-colored. Let wv_1^i be the outgoing $(j+1)$-colored edge at v_1^i and $v_k^i u$ be the outgoing j-colored edge at v_k^i. If $k > 1$, $v_1^i v_2^i$ is outgoing j-colored by definition. Thus, as $G[P_i]$ is induced, $w \notin P_i$. If $k = 1$, P_i consists of only one vertex and hence $w \notin P_i$. Thus, as $G[P_i]$ is a maximal j-$(j+1)$-colored path, wv_1^i is either unidirected $(j+1)$-colored or $(j+1)$-$(j+2)$-colored. As observed above, this implies $w \in V_{i-1}$ and by a $w \in C_{i-1}$. Similarly, we obtain $u \in C_{i-1}$.

Assume to the contrary that u is closer to r_{j+1} on C_{i-1} than w is. By definition of P_i, for every vertex of P_i, the outgoing j-colored edge is directed toward u and the outgoing $(j+1)$-colored edge points toward w on $G[P_i] \cup \{wv_1^i, v_k^i u\}$. By Definition 1c, the outgoing $(j+2)$-colored edge e of a vertex of P_i occurs in the counterclockwise sector from the outgoing j-colored to the outgoing $(j+1)$-colored edge excluding both. As we assumed that u is closer to r_{j+1} on C_{i-1} than w is, this sector is in the interior of the region bounded by $G[P_i] \cup \{wv_1^i, v_k^i u\}$ and the path from u to w on C_{i-1}. Hence, by planarity, e joins P_i with a vertex of $C_{i-1} \subseteq V_{i-1}$, contradicting our above characterization of edges that join P_i with vertices of V_{i-1}. Thus, w is closer to r_{j+1} on C_{i-1} than u is or we have $w = u$. If $u = w$, then Lemma 3 is violated by the cycle formed by $P_i \cup u$ in $T_j \cup T_{j+1}^{-1} \cup T_{j+2}^{-1}$, which is a contradiction. Thus, w is closer to r_{j+1} on C_{i-1} than u.

Since P_i is a maximal j-$(j+1)$-colored path, the outgoing j-colored and $(j+1)$-colored edges at every of its vertices are either in P_i or in $\{wv_1^i, v_k^i u\}$. Hence, by our above characterization, the edges that join P_i with vertices of $C_{i-1} \subseteq V_{i-1}$ are exactly $v_k^i u$, $v_1^i w$ and the unidirected incoming $(j+2)$-colored edges at vertices of P_i. Let vx be such an unidirected incoming $(j+2)$-colored edge with $v \in P_i$. By Definition 1c, vx occurs in the clockwise sector from the outgoing j-colored edge to the outgoing $(j+1)$-colored edge around v excluding both. By planarity and the fact that w is closer to r_{j+1} on C_{i-1} than u, x is contained in the path of C_{i-1} between w and u. By definition of the left and right neighbor v_0^i and v_{k+1}^i of P_i, we thus have $v_0^i = w$ and $v_{k+1}^i = u$, which proves b and c.

For d, let $v_l^i x \notin \{v_k^i v_{k+1}^i, v_1^i v_0^i\}$ be an edge that joins P_i with a vertex x of V_{i-1}. By a, $x \in C_{i-1}$. In the last paragraph, we observed that $v_l^i x$ is incoming $(j+2)$-colored at a vertex of P_i. We showed also that the outgoing j-colored and the outgoing $(j+1)$-edge of any vertex in P_i is either in P_i or $v_1^i v_0^i$ or $v_k^i v_{k+1}^i$. Thus, we obtain that $v_l^i x$ is unidirected incoming $(j+2)$-colored at a vertex of P_i. Assume, for the sake of contradiction, that $x = v_0^i$. Then the path from v_l^i to v_1^i on P_i, $v_0^i v_1^i$ and $v_l^i v_0^i$ form an oriented cycle in $T_j \cup T_{j+1}^{-1} \cup T_{j+2}^{-1}$, which contradicts Lemma 3. A similar argument shows $x \neq v_{k+1}^i$. ◀

3 Spanning Trees with Maximum Degree at Most 4

In this section, we prove our main result. The following new lemma on the structure of minimal Schnyder woods and their compatible ordered path partitions is crucial for this proof. For $0 < i \leq s$, let the path P_i *cover* an edge e or a vertex x if e or x is contained in C_{i-1}, but not in C_i, respectively.

▶ **Lemma 13.** *Let G be a σ-internally 3-connected plane graph, S be the minimal Schnyder wood of G^σ and $\mathcal{P}^{2,3} = (P_0, \dots, P_s)$ be the ordered path partition that is compatible with S. Let $P_i := \{v_1, \dots, v_k\} \neq P_0$ be a path of $\mathcal{P}^{2,3}$ and v_0 and v_{k+1} be its left and right neighbor. Then every edge $v_l w \notin \{v_0 v_1, v_k v_{k+1}\}$ with $v_l \in P_i$ and $w \in V_{i-1}$ is unidirected, 1-colored and incoming at v_k and $w \notin \{v_0, v_{k+1}\}$.*

Proof. Consider any edge $v_l w \notin \{v_0 v_1, v_k v_{k+1}\}$ that is incident to $v_l \in P_i$ and $w \in V_{i-1}$ (see Figure 3). By Lemma 12a, w is either v_0, v_{k+1} or a vertex that is covered by P_i. As $v_l w \notin \{v_0 v_1, v_k v_{k+1}\}$, $v_l w$ must be 1-colored incoming at v_l such that $w \notin \{v_0, v_{k+1}\}$ by Lemma 12d. It thus remains to show that $l = k$.

Assume to the contrary that $l \neq k$. Observe that, by Definition 1c, all edges in the clockwise sector from $v_l v_{l+1}$ to $v_l v_{l-1}$ are incoming 1-colored. Choose w such that $v_l w$ is the clockwise first incoming 1-colored edge at v_l (see Figure 3). By Corollary 6, the dual edge of $v_l v_{l+1}$ is unidirected 1-colored in the completion $\widetilde{G}_S$ of G and the dual edge of $v_l w$ is 2-3-colored. Hence, $\widetilde{G}_S$ contains the clockwise cycle shown in Figure 3, which contradicts the assumption that S is the minimal Schnyder wood. ◀

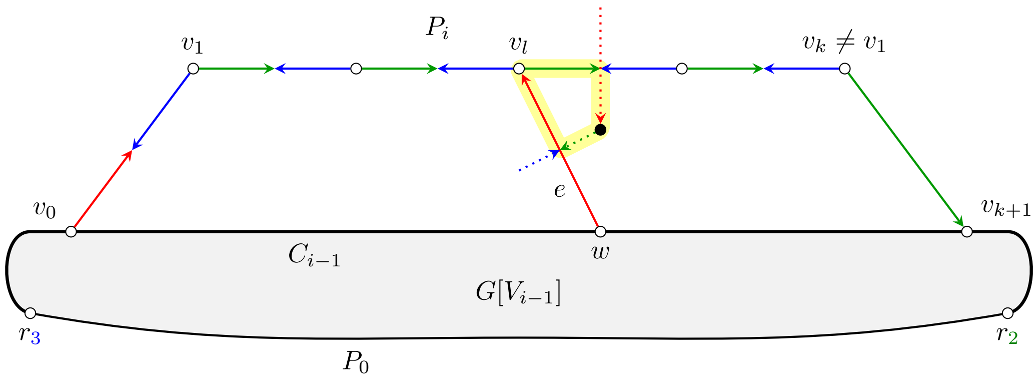


Figure 3 The clockwise cycle of $\widetilde{G}_S$ of the proof of Lemma 13, depicted in yellow.

For a spanning subgraph T of a plane graph G, let the *co-graph* $\neg T^*$ be the spanning subgraph $(V^*, (E(G) - E(T))^*)$ of G^*. As stated in the introduction, $\neg T^*$ is a spanning tree if T is one and in that case called a co-tree.

▶ **Theorem 14.** *Every $\{r_1, r_2, r_3\}$-internally 3-connected plane graph G contains a 4-tree T whose co-tree $\neg T^*$ is a 4-tree.*

Proof. We first sketch the general idea of the proof: First, we identify a spanning candidate graph $H \subseteq G$ such that $\neg H^*$ is a subgraph of G^* that has the same structural properties as H. We then define a subset D of the edges of H such that $H - D$ is acyclic and $\neg H^* + D^*$ has maximum degree 4. We use the same arguments to define a similar subset D' for $\neg H^*$. In the end, we need to show that D'^* and D^* do not create new cycles in $\neg H^*$ and H, respectively. That way we obtain that the co-graph of $H - D + D'^*$ is $\neg H^* - D' + D^*$, and both graphs are acyclic and of maximum degree 4. Since a spanning subgraph G' of G is connected if and only if $G - E(G')$ does not contain any edge cut of G, the cut-cycle duality [10, Prop. 4.6.1] proves that those two graphs are both connected, which gives the claim.

Let S be the minimal Schnyder wood of G^σ. By Lemma 7, the completion $\widetilde{G}_S$ of G contains no clockwise directed cycle. Since $\widetilde{G}_S$ contains the completion of the suspended dual G^{σ^*} except for its three outer vertices (which do not affect clockwise cycles), S^* is a minimal Schnyder wood of G^{σ^*}.

Let H be the spanning subgraph of G whose edge set consists of the bidirected edges of S. Recall that an edge $e \in E(G)$ is not in H if and only if e^* is in $\neg H^*$. By Definition 4, $\neg H^*$ contains therefore exactly the bidirected edges of S^*, except for the three bidirected edges on the outer face boundary of G^{σ^*}, as these are not dual edges of G (in fact, these three edges appear only in the suspended dual G^{σ^*} and were necessary to define dual Schnyder woods).

Since every vertex is incident to at most three bidirected edges by Definition 1c for S and as well for S^*, both H and $\neg H^*$ have maximum degree at most three. However, H and $\neg H^*$ may neither be connected nor acyclic. In fact, H contains always the outer face boundary of G as a cycle, as all edges are bidirected by the definition of the first paths of the compatible ordered path partitions $\mathcal{P}^{1,2}$, $\mathcal{P}^{2,3}$ and $\mathcal{P}^{3,1}$.

We will therefore iteratively identify edges of cycles of H such that $\neg H^*$ still has maximum degree at most four when those cycles are deleted in H. In order to do this, we iteratively define edge sets D and D' that are deleted from H and $\neg H^*$, starting with $D := D' := \emptyset$.

Let C be a cycle of H and let $(P_0, \dots, P_s)$ be the paths of the compatible ordered path partition $\mathcal{P}^{2,3}$ of S. Let P be the path of maximal length in C such that $P \subseteq P_M$ with $M := \max\{i \mid P_i \cap V(C) \neq \emptyset\}$; we call P the *index maximal subpath* of C, as it is the fraction of C highest up in the order of $\mathcal{P}^{2,3}$. Since C has only bidirected edges, the statement of Lemma 13 about e being unidirected implies that $P = P_M$ and that C contains the extension of P; in particular, $P \in \mathcal{P}^{2,3}$.

Denote by $\mathcal{P}_{max}$ the set of index maximal subpaths of all cycles of H. For a path $P \in \mathcal{P}_{max} \setminus \{P_s\}$, let P_L with $L := \min\{i \mid P_i$ covers an edge of the extension of $P\}$ be the *minimal-covering path* of P (recall that this extension is part of the cycle and the minimal-covering path exists, as P_s is excluded). Denote by $\mathcal{P}_{cover}$ the set of the minimal-covering paths of all index maximal subpaths in $\mathcal{P}_{max} \setminus \{P_s\}$. In particular, $P_s = r_1$ is the index maximal subpath of the outer face boundary of G, which is a bidirected cycle, as shown before. Since no edge of the extension of P_s is covered by another path of $\mathcal{P}^{2,3}$, we add the outgoing 2-colored edge of r_1 to D in order to destroy the outer face cycle.

Next, we process the paths of $\mathcal{P}_{cover}$ in reverse order of $\mathcal{P}^{2,3}$, i.e., from highest to lowest index. Let $P_c = \{v_1, \dots, v_k\} \in \mathcal{P}_{cover}$ for some $c \in \{1, \dots, s\}$ be the path under consideration. Let $P'_1, \dots, P'_l$ be the index maximal paths for which P_c is the minimal-covering path, ordered clockwise around the outer face of $G[V_{c-1}]$ (see Figure 4); note that there may also be other paths covered by P_c that are not index maximal. Let $f_1, \dots, f_a$ be the faces incident to v_k in counterclockwise order from the outgoing 3-colored edge to the outgoing 2-colored edge; we say that $f_1, \dots, f_a$ are *below* P_c. For every path of $\{P'_1, \dots, P'_l\}$, we will add an edge to D that is on the extension of that path. Thus, after having processed every path in $\mathcal{P}_{cover}$ in this way, a cycle in H does not exist in $H - D$ anymore.

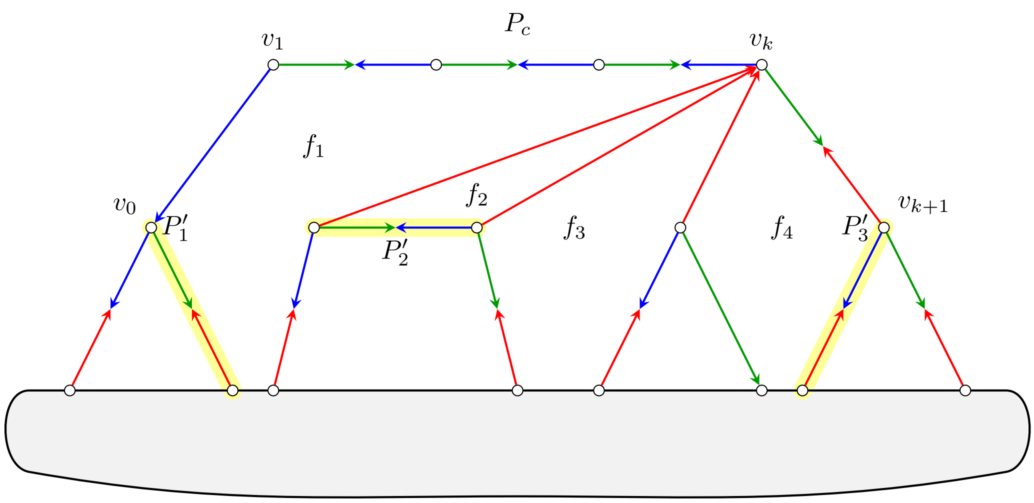


Figure 4 Illustration for some of the definitions used in the proof of Theorem 14. If Case 1 applies to P_c, we add the edges marked in yellow to D.

Consider the case that $v_{k+1} = w_1$ for a path $P'_l = \{w_1, \dots, w_t\}$. Assume for the sake of contradiction that then $v_k v_{k+1}$ is not 1-2-colored. Since P'_l is an index maximal subpath, $w_0 w_1$ is 1-3-colored. By Lemma 12c, then $v_k v_{k+1}$ is unidirected 2-colored. By Corollary 6,

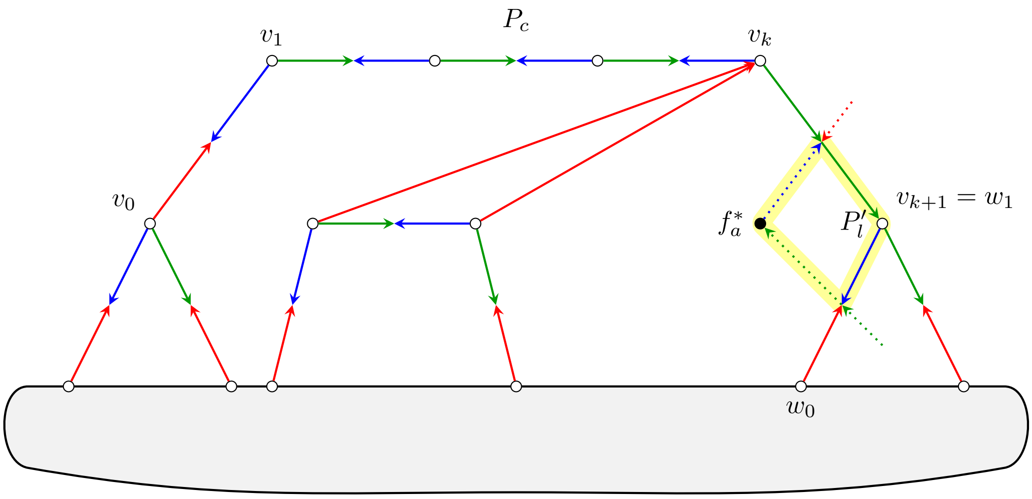


Figure 5 If $v_k v_{k+1}$ is unidirected 2-colored, then $\widetilde{G}_S$ contains the clockwise cycle depicted in yellow.

this implies that $(v_k v_{k+1})^*$ is 1-3-colored. Hence, $\widetilde{G}_S$ contains the clockwise cycle in Figure 5, which contradicts the assumption that S is the minimal Schnyder wood. We conclude that $v_k v_{k+1}$ is 1-2-colored in that case.

We will now select one edge from each of the extensions of the paths $P_1', \dots, P_l'$ and add it to D. We generally aim for selecting those edges that have smallest possible impact on the maximum degree of the dual graph: we prefer always edges of the paths $P_1', \dots, P_l'$ that are covered by P_c. For example, for P_2' in Figure 4, adding its edge to D causes a higher degree at the dual vertex f_2^* while connecting the dual to it; this is fine, as f_2 is a triangle by the mandatory outgoing 1-colored edges and thus the degree of f_2^* never exceeds 3 anyway. In detail, we distinguish the following two cases.

Augmentation procedure of D for the path P_c

Case 1: P_c is not an index maximal subpath (see Figure 4).

For every $i \in \{1, \dots, l\}$, if P_c covers an edge of $G[P_i']$, then we add one such edge to D. If for $P_l' = \{w_1, \dots, w_t\}$, we have $w_1 = v_{k+1}$ (note that this excludes the previous condition), then we add $w_0 w_1$ to D. For all remaining $i \in \{1, \dots, l\}$ for which none of the above conditions apply, we set $P_i' = \{u_1, \dots, u_t\}$ and add the edge $u_t u_{t+1}$ to D.

Case 2: P_c is an index maximal subpath.

Since the minimal-covering path of P_c has higher index than P_c itself, there already is either an edge of $G[P_c]$, $v_0 v_1$ or $v_k v_{k+1}$ in D.

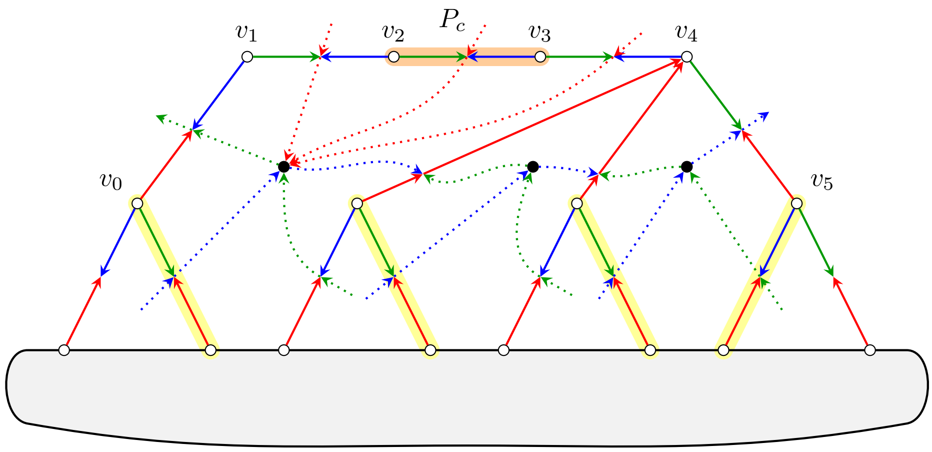
Case 2.1: An edge of $G[P_c]$ or $v_0 v_1$ is in D (see Figure 6a).

We proceed as in Case 1.

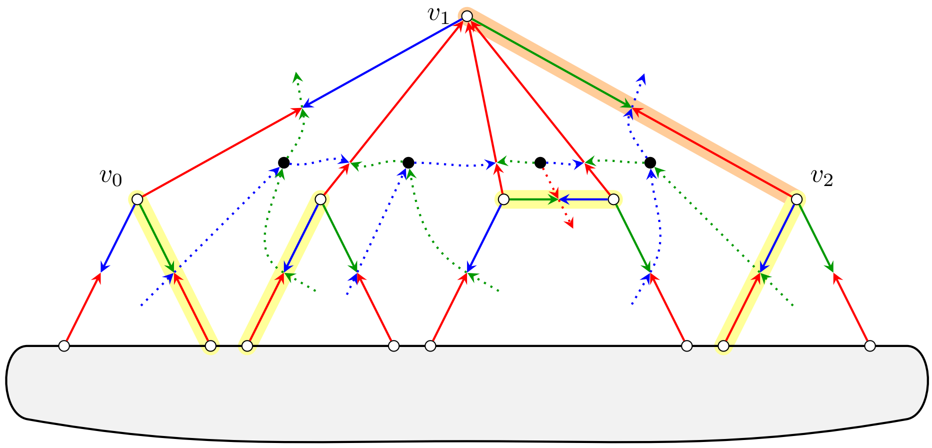
Case 2.2: $v_k v_{k+1} \in D$ (see Figure 6b)

For every $i \in \{1, \dots, l\}$, if P_c covers an edge of $G[P_i']$, then we add one such edge to D. If for $P_1' = \{p_1, \dots, p_b\}$, we have $p_b = v_0$ (note that this excludes the previous condition), then we add $p_b p_{b+1}$ to D. For all remaining $i \in \{1, \dots, l\}$ for which none of the above conditions apply, we set $P_i' = \{u_1, \dots, u_t\}$ and add the edge $u_0 u_1$ to D.

We now need to show that the maximum degree of $\neg H^* + D^*$ does not exceed 4. We prove that, after having processed P_c, no further boundary edge of any $f \in \{f_1, \dots, f_a\}$ is added to D: Assume to the contrary that there is a face $f \in \{f_1, \dots, f_a\}$ and an edge e



(a) The situation in Case 2.1. Here the edge v_2v_3 is marked in orange and in D before we consider P_c. The edges that we add to D are marked in yellow.



(b) The situation in Case 2.2. The edge v_1v_2 is marked in orange and in D before we consider P_c. The edges that we then add to D are marked in yellow.

Figure 6 Subcases for which P_c is an index maximal subpath in Theorem 14.

on the boundary of f such that e is not in D after having processed P_c but will be added later. Let $P_i \in \mathcal{P}^{2,3}$ be the path whose extension contains e. Then the minimal-covering path $P_{c'} \in \mathcal{P}^{2,3}$ of P_i needs to have lower index than P_c, i.e., $c' < c$. As e is covered by P_c, it is not covered by the minimal-covering path of P_i. Hence e will not be added to D, which is a contradiction.

First, consider the case $a > 1$, in which there at least two faces below P_c. By Definition 8b, the boundary of every f_j with $j \in \{1, \ldots, a\}$ contains at most two edges that are in the union of the extensions of paths in $\{P'_1, \ldots, P'_l\}$. For $j \in \{2, \ldots, a-1\}$, the augmentation procedure adds at most one of those edges to D, which implies that $\deg_{\neg H^*+D^*}(f^*_j) \leq 4$ for every $j \in \{2, \ldots, a-1\}$ (see Figure 6).

Now, consider $j = 1$, i.e., the face f_1 in the case $a > 1$. Let $P'_1 = \{p_1, \ldots, p_b\}$. In Case 1 of the augmentation procedure, we add at most one edge of the boundary of f_1 to D, hence $\deg_{\neg H^*+D^*}(f^*_1) \leq 4$. In Case 2, v_0v_1 is 1-3-colored, since P_c is an index maximal subpath (see Figure 6). By Corollary 6, $(v_0v_1)^*$ is unidirected 2-colored and outgoing at f^*_1. This implies $\deg_{\neg H^*}(f_1^*) \leq 2$, as f_1^* is incident to at most two bidirected edges. In Case 2.1, there is an edge of P_c or v_0v_1 in D. And if $p_b = v_0$, the edge p_bp_{b+1} is in D. Those are the only edges of the boundary of f_1 in D in Case 2.1 and hence $\deg_{\neg H^*+D^*}(f_1^*) \leq 4$. In Case 2.2, there is neither an edge of P_c nor v_0v_1 in D. As above, if $p_b = v_0$, the edge p_bp_{b+1} is in D. And if the left neighbor of P'_2 is no the boundary of f_1, then also the edge from P'_2 to its left neighbor is in D. Thus, also in Case 2.2, the augmentation procedure adds at most two edges of the boundary of f_1 to D and hence $\deg_{\neg H^*+D^*}(f_1^*) \leq 4$.

Now consider $j = a$, i.e., the face f_a in the case $a > 1$. Let $P'_l = \{w_1, \ldots, w_t\}$. If v_kv_{k+1} is 1-2-colored, then $(v_kv_{k+1})^*$ is unidirected 3-colored and outgoing at f_a^* by Corollary 6 and hence $\deg_{\neg H^*}(f_a^*) \leq 2$. The augmentation procedure adds at most two edges of the boundary of f_a to D and hence $\deg_{\neg H^*+D^*}(f_a^*) \leq 4$. Assume now that v_kv_{k+1} is unidirected 2-colored. Then P_c is not an index maximal subpath and we are in Case 1. As we observed above, then $w_1 \neq v_{k+1}$. The augmentation procedure adds at most one edge of the boundary of f_a to D and we have $\deg_{\neg H^*+D^*}(f_a^*) \leq 4$.

In the remaining case $a = 1$, there is exactly one face below P_c. If P_c is not an index maximal subpath, we use exactly the same arguments as we used to show that $\deg_{\neg H^*+D^*}(f_a^*) \leq 4$ for $a \neq 1$. If P_c is an index maximal subpath, then, by the same arguments as above, we know that $(v_kv_{k+1})^*$ and $(v_1v_0)^*$ are unidirected and outgoing at f_1^*. This implies $\deg_{\neg H^*}(f_1^*) \leq 1$. There at most three edges of the boundary of f_1 in D. Those potential edges are an edge of the extension of P_c, the outgoing 2-colored edge of v_0 and the outgoing 3-colored edge of v_{k+1}.

In addition, there are faces that are never below a path of $\mathcal{P}_{cover}$. Those faces have at most one edge of their boundary in D. Thus, their dual vertices in $\neg H^* + D^*$ have degree at most 4 (see Figure 6).

The clockwise path from r_2 to r_3 on the outer face boundary is not an index maximal subpath. Hence, the augmentation procedure does not add any edge of the clockwise path from r_2 to r_3 on the outer face boundary to D. However, by our assumption, D includes the outgoing 2-colored edge at r_1, which is the only edge of D that is on the boundary of the outer face of G.

So far we showed that $H - D$ is acyclic and $\neg H^* + D^*$ has maximum degree at most 4. We now apply the same arguments that we used for H to $\neg H^* \cup \{b_1b_2, b_2b_3, b_3b_1\}$ and obtain D'. Hence, we have that $\neg H^* \cup \{b_1b_2, b_2b_3, b_3b_1\} - D'$ is acyclic and $H + D'^* \setminus \{b_1b_2, b_2b_3, b_3b_1\}^*$ has maximum degree at most 4.

The edges b_1b_2, b_2b_3 and b_3b_1 are not in G^* and there is only one edge on the boundary of the outer face of G that is also in D. We may thus ignore b_1b_2, b_2b_3 and b_3b_1 in the following and freely switch from $\neg H^* \cup \{b_1b_2, b_2b_3, b_3b_1\}$ to $\neg H^*$. Hence, we also remove any of the edges b_1b_2, b_2b_3, b_3b_1 from D'.

Then the graphs $H - D + D'^*$ and $\neg H^* - D' + D^*$ have maximum degree at most 4 and by construction $\neg(H - D + D'^*)^* = \neg H^* - D' + D^*$. An edge set $E \subseteq E(G)$ is the edge set of a cycle in G if and only if the edge set E^* is a minimal edge cut in G^* [10, Prop. 4.6.1]. So in order to show that $\neg H^* - D' + D^*$ and $H - D + D'^*$ are both trees it suffices to show that they are both acyclic. We show that $\neg H^* - D' + D^*$ is acyclic. Applying the same arguments then shows that $H - D + D'^*$ is acyclic.

Assume to the contrary that there is a cycle C in $\neg H^* - D' + D^*$. Remember that for each index maximal subpath in $\mathcal{P}_{max}$ we pick exactly one edge of the extension and add it to D. This will finally lead to a contradiction. By construction, every cycle in $\neg H^*$ has at least one edge that is also in D'. Hence, C has at least one edge of D^*. Since every edge of D is in a cycle of H, by [10, Prop. 4.6.1], every edge in D^* joins two vertices of two different connected components of $\neg H^*$.

For a connected component K of $\neg H^*$, let $E_K \subseteq E(G^*)$ be the minimal edge cut separating K and $G^* - K$. Let C_K be the cycle of G with $E(C_K) = E_K^*$ and let $P^{C_K} = P_i \in \mathcal{P}^{2,3}$ be the index maximal subpath of C_K (see Figure 7). Choose K such that K shares a vertex with C and $P^{C_K} = P_i$ has smallest index. Since C is a cycle and intersects at least two connected components of $\neg H^*$, there are two edges $e, e' \in E_K$ that are also in C. Observe that these edges need to be in D^*.

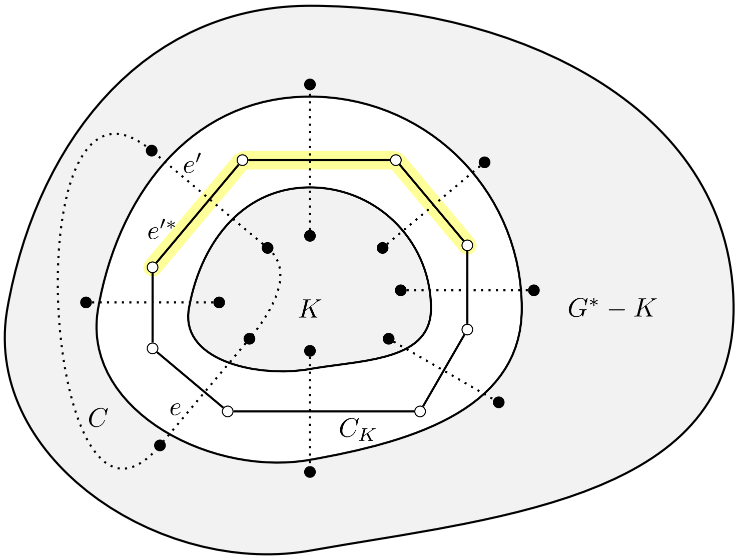


Figure 7 Illustration for the proof of Theorem 14. The extension of the path P^{C_K} is highlighted in yellow.

Then either e^* or e'^* is not in the extension of the index maximal subpath P^{C_K}. Assume w.l.o.g. that e^* is not in the extension of P^{C_K}. Let $P' = P_j \in \mathcal{P}^{2,3}$ for some $j \in \{1, \ldots, s\}$ be the path such that e^* is in the extension of P'. Since P^{C_K} is the index maximal subpath of C_K, we have $j < i$. So there exists a connected component K' of $\neg H^*$ such that K' and C have a vertex in common and P' is the index maximal subpath of the cycle $C_{K'}$ with $(E(C_{K'}))^*$ being the minimal cut separating K' and $G^* - K'$. This contradicts the definition of K. So $\neg H^* - D' + D^*$ and $H - D + D'^*$ are our desired trees. ◀

▶ **Corollary 15.** *Every 3-connected planar graph G contains a 4-tree T whose co-tree $\neg T^*$ is also a 4-tree.*

▶ **Corollary 16.** *The root r_1 is a leaf in $H - D + D'^*$ and all edges on the outer face of G except for the outgoing 2-colored edge at r_1 are in $H - D + D'^*$. We have $\deg_{H-D+D'^*}(r_3) = 2$ and $\deg_{H-D+D'^*}(r_2) \leq 3$. Also, the dual vertex of the outer face of G is a leaf in $\neg H^* - D' + D^*$.*

Proof. The proof of Theorem 14 yields that all edges on the outer face of G except for the outgoing 2-colored edge at r_1 are in $H - D + D'^*$. In $G^{\sigma *}$, the path $P_1 \in \mathcal{P}^{2,3}$ is given by the duals of the unidirected incoming 1-colored edges at r_1 (see Figure 2). Since the outgoing 2-colored and the outgoing 3-colored edge at r_1 are bidirected, P_1 is not an index maximal subpath and hence none of the duals of the unidirected incoming 1-colored edges at r_1 is added to D'. Thus, r_1 is a leaf in $H - D + D'^*$.

The dual edges of the incoming unidirected edges at r_2 and r_3 are all covered by the last singleton b_1 of $\mathcal{P}^{2,3}$ of $\neg H^* \cup \{b_1b_2, b_2b_3, b_3b_1\}$ (see Figure 2). Let e_2 be the dual of the clockwise first unidirected 2-colored incoming edge at r_2 and e_3 be the dual of the counterclockwise first unidirected 3-colored incoming edge at r_3. Let I_i be the set of the duals of the unidirected i-colored incoming edges at r_i, $i = 2, 3$. For $e \in I_i$, $i = 2, 3$ let $P_e \in \mathcal{P}^{2,3}$ be the path such that e belongs to the extension of P_e. Observe that, for all edges $e \in (I_2 \setminus \{e_2\}) \cup (I_3 \setminus \{e_3\})$, b_1 is not the minimal-covering path of P_e. Hence, those edges are not added to D'. On the other hand b_1 might be the minimal-covering path of P_{e_2} and/or P_{e_3}. Since we added b_1b_2 to D', we do not add e_3 to D' but might do so for e_2 (compare Case 2.2 in the proof of Theorem 14). Hence, $\deg_{H-D+D'^*}(r_3) = 2$ and $\deg_{H-D+D'^*}(r_2) \leq 3$.

Since the outgoing 2-colored edge at r_1 is the only edge on the boundary of the outer face f that is not in $H - D + D'^*$, we know that the vertex f^* is a leaf in $\neg H^* - D' + D^*$. ◀

4 Relaxing Connectivity Assumptions

In this section, we relax the connectivity condition. A common relaxation of σ-internal 3-connectedness is internal 3-connectedness. A plane graph G is *internally 3-connected* if adding a vertex, the *apex vertex*, in the outer face and connecting this new vertex with all the vertices on the outer face of G results in a 3-connected graph. Observe that every σ-internally 3-connected graph is also internally 3-connected.

▶ Remark 17. The statement of Theorem 14 does not hold for internally 3-connected graphs. There exist internally 3-connected plane graphs G_k on $2k$ vertices such that every spanning tree of the dual graph has maximum degree at least $\lceil k/2 \rceil$.

Proof. In order to define G_k, fix an embedding of the cycle C_k on k vertices. Let $w_0, \ldots, w_{k-1}$ be the vertices of this cycle in clockwise order. For every $i = 0, \ldots, k-1$, add a vertex p_i in the outer face and add edges p_iw_i and p_iw_{i+1} (indices taken modulo k) such that the resulting graph G_k is still plane (see Figure 8). Clearly, G_k is internally 3-connected. The dual of G_k contains parallel edges. Its underlying graph, in which all those vertex pairs joined by parallel edges are only joined by one edge, is the complete bipartite graph $K_{2,k}$. By pigeonhole principle, every spanning tree of $K_{2,k}$ has maximum degree at least $\lceil k/2 \rceil$. ◀

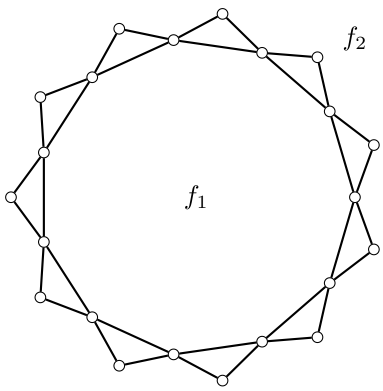


Figure 8 The graph G_{11} of Remark 17. In every spanning tree of the dual graph, f_1^* or f_2^* has degree at least 6.

However, we can apply Theorem 14 to $G + x$ for an internally 3-connected graph G with an apex vertex x. Then, we obtain after small modifications a 4-tree of G and a tree of G^* such that all vertices except for the dual of the outer face have degree at most 4. This motivates the notion of *k-internally 3-connected* graphs. G is *k-internally 3-connected* if there are k vertices $w_1, \dots, w_k$ on the outer face of G such that adding an apex vertex x in the outer face and the edges xw_i for all $i \in \{1, \dots, k\}$ yields a 3-connected graph. Observe that every σ-internally 3-connected graph is k-internally 3-connected for $k \geq 3$ and every k-internally 3-connected graph is also internally 3-connected.

▶ **Lemma 18.** *For every k-internally 3-connected plane graph G there exists a 4-tree such that all vertices of its co-tree except for the dual of the outer face have degree at most 4. The dual of the outer face has degree at most $2k-2$.*

Proof. Let G^x be the plane graph obtained by adding and connecting the apex vertex as described in the statement. Define $r_1 := x$ and r_2 and r_3 to be its clockwise and counterclockwise neighbor on the outer face of G^x, respectively. Let $w_1, \dots, w_k$ be ordered clockwise around the outer face of G such that $w_1 = r_3$, $w_k = r_2$ and $w_i x \in E(G^x)$ for all $i \in \{1, \dots, k\}$ (Figure 9). We now apply Theorem 14 to G^x with this choice of roots. We obtain a 4-tree T of G^x such that $\neg T^*$ is a 4-tree of G^{x*}. Observe that by Corollary 16 all edges on the outer face of G^x except for $r_1 r_2$ are in T, $\deg_T(r_1) = 1$ and $\deg_T(r_3) = 2$ (see Figure 9). Thus, we have that $w_1 x \in E(T)$ and $w_i x \notin E(T)$ for all $i \in \{2, \dots, k\}$. Hence, $T - w_1 x$ is a 4-tree of G. We consider the dual graph. As $T - w_1 x$ is a 4-tree of G, its co-tree is also a spanning tree of G^*. As $\neg T^*$ is a 4-tree of G^{x*}, we obtain that in the co-tree of $T - w_1 x$ every vertex except for the dual of the outer face has degree at most 4.

We consider the outer face. Take a Schnyder wood as in the proof of Theorem 14. Let d_i be the dual vertex of the face incident to $w_i x$ and $w_{i+1} x$ for $i \in \{1, \dots, k-1\}$ in G^x. In $\neg T^*$, those vertices have degree at most 4. Consider d_1. The dual edge $(r_1 r_3)^*$ is outgoing at d_1. The edge e preceding $r_1 r_3$ on the face d_1^* in clockwise order is unidirected 3-colored and incoming at r_3. Thus, there is no index maximal subpath that contains e. And hence, in the algorithm of the proof of Theorem 14, we add at most one edge to D that is incident to d_1. Furthermore, $(xw_2)^*$ is incident to d_1 and in $E(\neg T^*)$. Therefore, there are at most two edges on the clockwise path from r_3 to w_2 on the outer face of G that are not in T.

As $(w_i x)^*$ and $(w_{i+1} x)^*$ are incident to d_i and $(w_i x)^*, (w_{i+1} x)^* \in E(\neg T^*)$ for all $i \in \{2, \dots, k-1\}$, we obtain, that there are at most two edges on the clockwise path from w_i to w_{i+1} on the outer face of G that are not in T. And hence, the dual vertex of the outer face of G has degree at most $2k-2$ in the co-tree of $T - w_1 x$. ◀

5 Computational Aspects

Let G^σ be the suspension of a σ-internally 3-connected plane graph and let S be the minimal Schnyder wood of G^σ. Badent et al. showed that an ordered path partition $\mathcal{P}^{2,3}$ that is compatible to S can be computed in time $O(n)$ [2, Theorem 7]. This $\mathcal{P}^{2,3}$ can also be used to compute S itself in the same time [2, Theorem 5], which in turn allows to compute the dual S^* and thus also the candidate graphs H and $\neg H^*$in linear time.

For $i := 1, \dots, s$, we detect whether $H \cap G[V_i]$ has a cycle that contains the extension of P_i by maintaining the connected components of the previous graph $H \cap G[V_{i-1}]$ and querying whether the left and right neighbor of P_i are in the same connected component of $H \cap G[V_{i-1}]$. This can be done in amortized constant time per step using the special union-find data structure in [16], since the structure of possible union operations is a tree. This gives the set $\mathcal{P}_{max}$ of all index maximal subpaths in $\mathcal{P}^{2,3}$ and their minimial-covering paths.

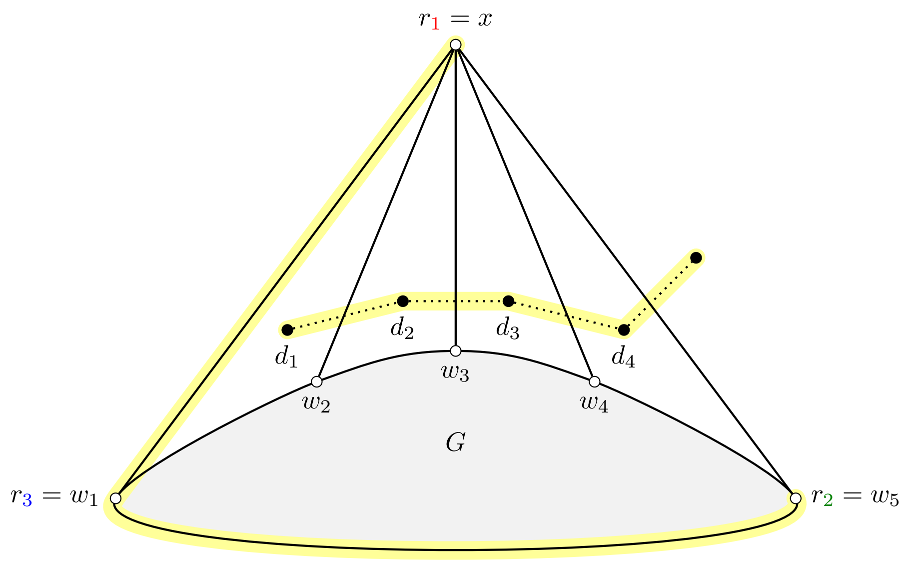


Figure 9 Situation as in Lemma 18. A 5-internally 3-connected graph G with its apex vertex x. Some edges of the 4-tree T of G^x and its co-tree are highlighted in yellow.

Since the case distinction and every step of the augmentation procedure for every minimal-covering path P_c can be computed in constant time per index-maximal subpath, we obtain an algorithm with running time $O(n)$ to compute a 4-tree of G whose co-tree is also a 4-tree.

6 Conclusion

We used Schnyder woods in order to prove that every (σ-)internally 3-connected graph has a 4-tree such that its co-tree is also a 4-tree. Also, we showed that there is a linear time algorithm computing such a tree. If we further relax the connectivity condition to (k-)internal 3-connectedness, then we cannot expect a 4-tree on the dual anymore. However, we always manage to find a tree such that at most one vertex of its co-tree has degree larger than 4.

Grünbaum's conjecture still remains open. We believe that it could prove worthwhile to assume further restrictions on the graph in order to decrease the maximum degree in both the tree and its co-tree or only one of them.

References

1 Md. J. Alam, W. Evans, S. G. Kobourov, S. Pupyrev, J. Toeniskoetter, and T. Ueckerdt. Contact representations of graphs in 3D. In *Proceedings of the 14th International Symposium on Algorithms and Data Structures (WADS '15)*, volume 9214 of *Lecture Notes in Computer Science*, pages 14–27, 2015. Technical Report accessible on arXiv: `arXiv:1501.00304`. `doi:10.1007/978-3-319-21840-3_2`.

2 M. Badent, U. Brandes, and S. Cornelsen. More canonical ordering. *Journal of Graph Algorithms and Applications*, 15(1):97–126, 2011. `doi:10.7155/JGAA.00219`.

3 D. Barnette. Trees in polyhedral graphs. *Canadian Journal of Mathematics*, 18:731–736, 1966. `doi:10.4153/CJM-1966-073-4`.

4 D. W. Barnette. 2-Connected spanning subgraphs of planar 3-connected graphs. *Journal of Combinatorial Theory, Series B*, 61:210–216, 1994. `doi:10.1006/jctb.1994.1045`.

5 T. Biedl. Trees and co-trees with bounded degrees in planar 3-connected graphs. In *14th Scandinavian Symposium and Workshops on Algorithm Theory (SWAT'14)*, pages 62–73, 2014. doi:10.1007/978-3-319-08404-6_6.

6 T. Böhme, J. Harant, M. Kriesell, S. Mohr, and J. M. Schmidt. Rooted minors and locally spanning subgraphs. *Journal of Graph Theory*, 105(2):209–229, 2024. doi:10.1002/jgt.23012.

7 R. Brunet, M. N. Ellingham, Z. Gao, A. Metzlar, and R. B. Richter. Spanning planar subgraphs of graphs in the torus and Klein bottle. *Journal of Combinatorial Theory, Series B*, 65(1):7–22, 1995. doi:10.1006/jctb.1995.1041.

8 P. O. de Mendez. *Orientations bipolaires.* PhD thesis, École des Hautes Études en Sciences Sociales, Paris, 1994.

9 G. Di Battista, R. Tamassia, and L. Vismara. Output-sensitive reporting of disjoint paths. *Algorithmica*, 23(4):302–340, 1999. doi:10.1007/PL00009264.

10 R. Diestel. *Graph theory.* Graduate texts in mathematics 173. Springer, Berlin, 4th edition edition, 2012. URL: http://swbplus.bsz-bw.de/bsz377230375cov.htm.

11 S. Felsner. Geodesic embeddings and planar graphs. *Order*, 20:135–150, 2003.

12 S. Felsner. *Geometric Graphs and Arrangements.* Advanced Lectures in Mathematics. Vieweg+Teubner, Wiesbaden, 2004. doi:10.1007/978-3-322-80303-0.

13 S. Felsner. Lattice structures from planar graphs. *Electronic Journal of Combinatorics*, 11(1):R15, 1–24, 2004. doi:10.37236/1768.

14 Stefan Felsner. Convex drawings of planar graphs and the order dimension of 3-polytopes. *Order*, 18(1):19–37, 2001. doi:10.1023/A:1010604726900.

15 É. Fusy. *Combinatorics of planar maps and algorithmic applications (Combinatoire des cartes planaires et applications algorithmiques).* PhD thesis, École Polytechnique, Palaiseau, France, 2007. URL: https://tel.archives-ouvertes.fr/pastel-00002931.

16 H. N. Gabow and R. E. Tarjan. A linear-time algorithm for a special case of disjoint set union. *Journal of Computer and System Sciences*, 30(2):209–221, 1985.

17 Z. Gao. 2-Connected coverings of bounded degree in 3-connected graphs. *Journal of Graph Theory*, 20(3):327–338, 1995. doi:10.1002/JGT.3190200309.

18 Z. Gao and R. B. Richter. 2-Walks in circuit graphs. *Journal of Combinatorial Theory, Series B*, 62(2):259–267, 1994. doi:10.1006/JCTB.1994.1068.

19 B. Grünbaum. Polytopes, graphs, and complexes. *Bulletin of the American Mathematical Society*, 76(6):1131–1201, 1970. doi:10.1090/S0002-9904-1970-12601-5.

20 G. Kant. Drawing planar graphs using the lmc-ordering. In *Proceedings of the 33rd Annual Symposium on Foundations of Computer Science (FOCS'92)*, pages 101–110, 1992. doi:10.1109/SFCS.1992.267814.

21 G. Kant. Drawing planar graphs using the canonical ordering. *Algorithmica*, 16(1):4–32, 1996. doi:10.1007/BF02086606.

22 K. Ota and K. Ozeki. Spanning trees in 3-connected $K_{3,t}$-minor-free graphs. *Journal of Combinatorial Theory, Series B*, 102:1179–1188, 2012. doi:10.1016/J.JCTB.2012.07.002.

23 K. Ozeki and T. Yamashita. Spanning trees: A survey. *Graphs and Combinatorics*, 27:1–26, 2011. doi:10.1007/S00373-010-0973-2.

24 W. Schnyder. Embedding planar graphs on the grid. In *Proceedings of the 1st Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 138–148, 1990. URL: http://dl.acm.org/citation.cfm?id=320176.320191.