Toward Grünbaum's Conjecture for 4-Connected Graphs

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Abstract

Given a spanning tree T of a 3-connected planar graph G, the *co-tree* of T is the spanning tree of the dual graph G^* given by the duals of the edges that are not in T. Grünbaum conjectured in 1970 that there is such a spanning tree T such that T and its co-tree both have maximum degree at most 3.

In 2014, Biedl proved that there is a spanning tree T such that T and its co-tree have maximum degree at most 5. Using structural insights into Schnyder woods, Schmidt and the author recently improved this bound on the maximum degree to 4. In this paper, we prove that in a 4-connected planar graph there exists a spanning tree T of maximum degree at most 3 such its co-tree has maximum degree at most 4. This almost solves Grünbaum's conjecture for 4-connected graphs.

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1 Introduction

In 1966, Barnette showed that every 3-connected planar graph has a spanning 3-tree [3]. Here and in the following, a k-tree denotes a tree with maximum degree at most k. The dual of a 3-connected planar graph G is also 3-connected and planar. As mentioned in the abstract, for a spanning tree T the duals of the edges $E(G) - E(T)$ form a spanning tree of the dual graph G^*, the so called co-tree $\neg T^*$ of T. Thus, the question of a simultaneous bound on the maximum degree of T and its co-tree naturally arises. In 1970, Grünbaum made the following conjecture.

▶ **Conjecture 1** (Grünbaum [16, p. 1148], 1970). *Every planar 3-connected graph G contains a 3-tree T whose co-tree $\neg T^*$ is also a 3-tree.*

While Grünbaum's conjecture is to the best of our knowledge still unsolved, progress has been made by Biedl [4], who proved the existence of a 5-tree whose co-tree is a 5-tree. Her approach uses structural properties of canonical orderings. Schmidt and the author recently proved that in a 3-connected planar graph there is a spanning tree T such that T and its co-tree have maximum degree at most 4 [20]. In this paper, we prove that in a 4-connected planar graph there is a spanning 3-tree such that its co-tree is a spanning 4-tree of the dual. We use structural properties of minimal Schnyder woods. Schnyder woods are a tool which is widely applied in graph drawing [1, 13, 15, 21, 22] and beyond [6, 7, 10, 17].

Our approach divides into two steps. Let G be the graph in question. First, we identify a suitable candidate graph H. This candidate is a spanning and connected subgraph of G of maximum degree at most 3. By the well-known cut-cycle duality [11, Prop. 4.6.1], its co-graph (that is the graph with edge set $E(G^*) - E(H)^*$) is acyclic. We show that the co-graph also has maximum degree 3. Then, we only lack the acyclicity of the primal graph.

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Thus, in the second step, we delete edges of the candidate graph H such that it becomes acyclic and the co-graph remains acyclic. Then, we show that the degree of the resulting co-graph does not exceed 4. This then yields the desired 3-tree such that its co-tree is a 4-tree. The first step is far from being trivial. Especially, it is hard to prove the connectivity of the candidate graph. The second step uses parts of the proof of the main theorem of [20]. Hence, we focus on the first step.

We discuss Schnyder woods, their lattice structure and ordered path partitions in Section 2, the candidate graph H in Section 3 and the second and final step in Section 4. Due to space limitations some proofs are omitted or only sketched.

2 Schnyder Woods and Ordered Path Partitions

We only consider simple undirected graphs. A graph is *plane* if it is planar and embedded into the Euclidean plane without intersecting edges. The *neighborhood of a vertex set A* is the union of the neighborhoods of vertices in A. Although parts of this paper use orientation on edges, we will always let vw denote the undirected edge $\{v, w\}$.

2.1 Schnyder Woods

Let $\sigma := \{r_1, r_2, r_3\}$ be a set of three vertices of the outer face boundary of a plane graph G in clockwise order (but not necessarily consecutive). We call r_1, r_2 and r_3 *roots*. The *suspension G^σ* of G is the graph obtained from G by adding at each root of σ a half-edge pointing into the outer face. With a little abuse of notation, we define a *half-edge* as an arc that has a startvertex but no endvertex.

▶ **Definition 2** (Felsner [12]). *Let $\sigma = \{r_1, r_2, r_3\}$ and G^σ be the suspension of a 3-connected plane graph G. A* Schnyder wood *of G^σ is an orientation and coloring of the edges of G^σ (including the half-edges) with the colors 1,2,3 (red, green, blue) such that*

(a) *Every edge e is oriented in one direction (we say e is* unidirected*) or in two opposite directions (we say e is* bidirected*). Every direction of an edge is colored with one of the three colors 1,2,3 (we say an edge is i-*colored *if one of its directions has color i) such that the two colors i and j of every bidirected edge are distinct (we call such an edge i-j-*colored*). Throughout the paper, we assume modular arithmetic on the colors 1,2,3 in such a way that $i+1$ and $i-1$ for a color i are defined as $(i \mod 3)+1$ and $(i+1 \mod 3)+1$, respectively. For a vertex v, a uni- or bidirected edge is* incoming *(i-colored)* in *v if it has a direction (of color i) that is directed toward v, and* outgoing *(i-colored)* of *v if it has a direction (of color i) that is directed away from v.*

(b) *For every color i, the half-edge at r_i is unidirected, outgoing and i-colored.*

(c) *Every vertex v has exactly one outgoing edge of every color. The outgoing 1-, 2-, 3-colored edges e_1, e_2, e_3 of v occur in clockwise order around v. For every color i, every incoming i-colored edge of v is contained in the clockwise sector around v from e_{i+1} to e_{i-1} (see Figure 1).*

(d) *No inner face boundary contains a directed cycle (disregarding possible opposite edge directions) in one color.*

For a Schnyder wood and color i, let T_i be the directed graph that is induced by the directed edges of color i. The following result justifies the name of Schnyder woods.

▶ **Lemma 3** ([13, 21]). *For every color i of a Schnyder wood of G^σ, T_i is a directed spanning tree of G in which all edges are oriented to the root r_i.*

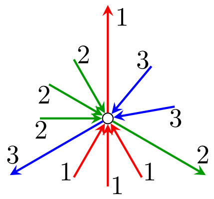


Figure 1 Properties of Schnyder woods. Condition 2c at a vertex.

For a directed graph H, we denote by H^{-1} the graph obtained from H by reversing the direction of all its edges.

▶ **Lemma 4** (Felsner [15]). *For every* $i \in \{1, 2, 3\}$, $T_i^{-1} \cup T_{i+1}^{-1} \cup T_{i+2}$ *is acyclic.*

▶ **Lemma 5** (folklore). *Let* S *be a Schnyder wood of* G^σ. *Then,* G *is internally triangulated, i.e., every face except the outer face is a triangle if and only if every internal edge of G is unidirected in* S.

2.2 Dual Schnyder Woods

Let G be a 3-connected plane graph. Any Schnyder wood of G^σ induces a Schnyder wood of a slightly modified planar dual of G^σ in the following way [9, 14] (see [19, p. 30] for an earlier variant of this result given without proof). As common for plane duality, we will use the plane dual operator * to switch between primal and dual objects (also on sets of objects).

Extend the three half-edges of G^σ to non-crossing infinite rays and consider the planar dual of this plane graph. Since the infinite rays partition the outer face f of G into three parts, this dual contains a triangle with vertices b_1, b_2 and b_3 instead of the outer face vertex f^* such that b_i^* is not incident to r_i for every i (see Figure 2). Let the *suspended dual* G^{σ^*} of G^σ be the graph obtained from this dual by adding at each vertex of $\{b_1, b_2, b_3\}$ a half-edge pointing into the outer face.

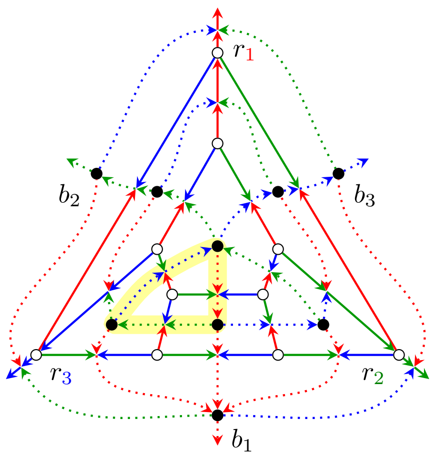


Figure 2 The completion of G obtained by superimposing G^σ and its suspended dual G^{σ^*} (the latter depicted with dotted edges). The primal Schnyder wood is not the minimal element of the lattice of Schnyder woods of G, as this completion contains a clockwise directed cycle (marked in yellow).

Consider the superposition of G^σ and its suspended dual G^{σ^*} such that exactly the primal dual pairs of edges cross (here, for every $1 \leq i \leq 3$, the half-edge at r_i crosses the dual edge $b_{i-1}b_{i+1}$).

▶ **Definition 6.** *For any Schnyder wood S of G^σ, define the orientation and coloring S^* of the suspended dual G^{σ^*} as follows (Figure 2):*

(a) *For every unidirected $(i-1)$-colored edge or half-edge e of G^σ, color e^* with the two colors i and $i+1$ such that e points to the right of the i-colored direction.*

(b) *Vice versa, for every i-$(i+1)$-colored edge e of G^σ, $(i-1)$-color e^* unidirected such that e^* points to the right of the i-colored direction.*

(c) *For every color i, make the half-edge at b_i unidirected, outgoing and i-colored.*

The following lemma states that S^* is indeed a Schnyder wood of the suspended dual. The vertices b_1, b_2 and b_3 are called the *roots* of S^*.

▶ **Lemma 7** ([18], [14, Prop. 3])**.** *For every Schnyder wood S of G^σ, S^* is a Schnyder wood of G^{σ^*}.*

Since $S^{*^*} = S$, Lemma 7 gives a bijection between the Schnyder woods of G^σ and the ones of G^{σ^*}. Let the *completion* $\widetilde{G}$ of G be the plane graph obtained from the superposition of G^σ and G^{σ^*} by subdividing each pair of crossing (half-)edges with a new vertex, which we call a *crossing vertex* (Figure 2). The completion has six half-edges pointing into its outer face.

Any Schnyder wood S of G^σ implies the following natural orientation and coloring $\widetilde{G}_S$ of its completion $\widetilde{G}$: For any edge $vw \in E(G^\sigma) \cup E(G^{\sigma^*})$, let z be the crossing vertex of $\widetilde{G}$ that subdivides vw and consider the coloring of vw in either S or S^*. If vw is outgoing of v and i-colored, we direct $vz \in E(\widetilde{G})$ toward z and i-color it; analogously, if vw is outgoing of w and j-colored, we direct $wz \in E(\widetilde{G})$ toward z and j-color it. In the case that vw is unidirected, say w.l.o.g. incoming at v and i-colored, we direct $zv \in E(\widetilde{G})$ toward v and i-color it. The three half-edges of G^{σ^*} inherit the orientation and coloring of S^* for $\widetilde{G}_S$. By Definition 6, the construction of $\widetilde{G}_S$ implies immediately the following corollary.

▶ **Corollary 8.** *Every crossing vertex of $\widetilde{G}_S$ has one outgoing edge and three incoming edges and the latter are colored 1, 2 and 3 in counterclockwise direction.*

Using results on orientations with prescribed outdegrees on the respective completions, Felsner and Mendez [8, 13] showed that the set of Schnyder woods of a planar suspension G^σ forms a distributive lattice. The order relation of this lattice relates a Schnyder wood of G^σ to a second Schnyder wood if the former can be obtained from the latter by reversing the orientation of a directed clockwise cycle in the completion. This gives the following lemma.

▶ **Lemma 9** ([8, 13])**.** *For the minimal element S of the lattice of all Schnyder woods of G^σ, $\widetilde{G}_S$ contains no clockwise directed cycle.*

We call the minimal element of the lattice of all Schnyder woods of G^σ the *minimal Schnyder wood* of G^σ.

▶ **Lemma 10** (Di Battista et al. [9])**.** *The boundary of every internal face of G can be partitioned into six paths $P_{1,3}$, $p_{2,3}$, $P_{2,1}$, $p_{3,1}$, $P_{3,2}$ and $p_{1,2}$ which appear in that clockwise order. For those paths the following holds (see Figure 3).*

(a) *$P_{i,j}$ consists of one edge which is either unidirected i-colored, unidirected j-colored or i-j-colored. Color i is directed in clockwise direction and color j in counterclockwise direction around f.*

(b) *$p_{i,j}$ consists of a possibly empty sequence of i-j-colored edges such that color i is directed clockwise around f.*

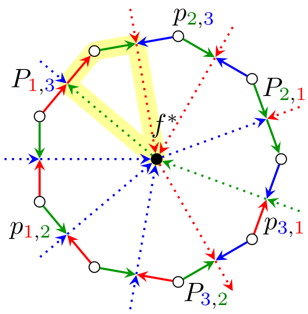


Figure 3 Illustration for Lemma 10. A face f, the paths on its boundary and the dual edges incident to f^*. If $P_{1,3}$ is unidirected 1-colored and $p_{2,3}$ is non-empty, there is a clockwise cycle in $\widetilde{G}_S$, marked in yellow.

2.3 Ordered Path Partitions

We denote paths as tuples of vertices such that consecutive vertices in the tuple are adjacent in the path. If a path P consists of only one vertex x, we might also write $P = x$. The concatenation of two paths P_1 and P_2 we denote by P_1P_2.

▶ **Definition 11.** *For any $j \in \{1, 2, 3\}$ and any $\{r_1, r_2, r_3\}$-internally 3-connected plane graph G, an* ordered path partition *$\mathcal{P} = (P_0, \dots, P_s)$ of G* with base-pair *(r_j, r_{j+1}) is a tuple of induced paths such that their vertex sets partition $V(G)$ and the following holds for every $i \in \{0, \dots, s-1\}$, where $V_i := \bigcup_{q=0}^{i} V(P_q)$ and the* contour *C_i is the clockwise walk from r_{j+1} to r_j on the outer face of $G[V_i]$.*

(a) *P_0 is the clockwise path from r_j to r_{j+1} on the outer face boundary of G, and $P_s = r_{j+2}$.*
(b) *Every vertex in P_i has a neighbor in $V(G) \setminus V_i$.*
(c) *C_i is a path.*
(d) *Every vertex in C_i has at most one neighbor in P_{i+1}.*

By Definition 11a and 11b, G contains for every i and every vertex $v \in P_i$ a path from v to r_{j+2} that intersects V_i only in v. Since G is plane, we conclude the following.

▶ **Lemma 12.** *Every path P_i of an ordered path partition is embedded into the outer face of $G[V_{i-1}]$ for every $1 \le i \le s$.*

2.3.1 Compatible Ordered Path Partitions

We describe a connection between Schnyder woods and ordered path partitions that was first given by Badent et al. [2, Theorem 5] and then revisited by Alam et al. [1, Lemma 1].

▶ **Definition 13.** *Let $j \in \{1, 2, 3\}$ and S be any Schnyder wood of the suspension G^σ of G. As proven in [1, arXiv version, Section 2.2], the inclusion-wise maximal j-$(j+1)$-colored paths of S then form an ordered path partition of G with base pair (r_j, r_{j+1}), whose order is a linear extension of the partial order given by reachability in the acyclic graph $T_j^{-1} \cup T_{j+1}^{-1} \cup T_{j+2}$; we call this special ordered path partition* compatible *with S and denote it by $\mathcal{P}^{j,j+1}$.*

For example, for the Schnyder wood given in Figure 2, $\mathcal{P}^{2,3}$ consists of the six maximal 2-3-colored paths, of which four are single vertices. We denote each path $P_i \in \mathcal{P}^{j,j+1}$ by $P_i := (v_1^i, \dots, v_k^i)$ such that $v_1^i v_2^i$ is outgoing j-colored at v_1^i.

Let C_i be as in Definition 11. By Definition 11c and Lemma 12, every path $P_i = (v_1^i, \dots, v_k^i)$ of an ordered path partition satisfying $i \in \{1, \dots, s\}$ has a neighbor $v_0^i \in C_{i-1}$ that is closest to r_{j+1} and a different neighbor $v_{k+1}^i \in C_{i-1}$ that is closest to r_j. We call v_0^i the *left neighbor* of P_i, v_{k+1}^i the *right neighbor* of P_i and $P_i^e := v_0^i P_i v_{k+1}^i$ the *extension* of P_i; we omit superscripts if these are clear from the context. For $0 < i \leq s$, let the path P_i *cover* an edge e or a vertex x if e or x is contained in C_{i-1}, but not in C_i, respectively.

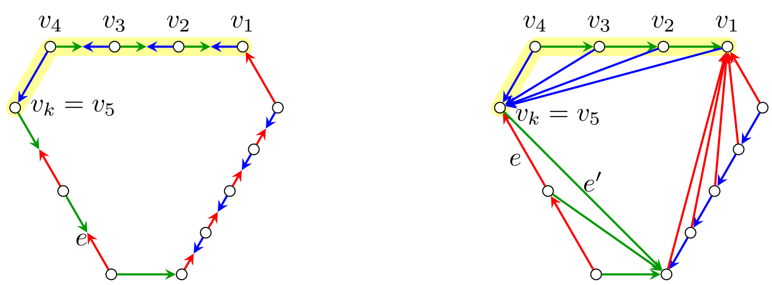
3 The Candidate Graph H

In this section, we define a special triangulation $\tau(G)$ of G. We also give some structural properties of $\tau(G)$. Afterwards, we define our candidate graph H as a subgraph of $\tau(G)$. Then, we are left to show some structural properties of H. First, we show that H is also a subgraph of G. Then, we argue that H and its co-graph have maximum degree at most 3. And finally, we show that if G is 4-connected, then H is connected. The latter will need some technical preparation.

Let P be the counterclockwise 3-colored path on the boundary of some internal face. By Lemma 10, P consists of $p_{2,3}$ (a possibly empty sequence of 2-3-colored edges) and possibly $P_{1,3}$ (an edge which is either unidirected 1-colored, unidirected 3-colored or 1-3-colored). Since S is minimal, we do not have clockwise cycles in $\widetilde{G}_S$. Hence, if $p_{2,3}$ is non-empty, then $P_{1,3}$ is either unidirected 3-colored or 3-1-colored (Figure 3), and we might define $\tau(G)$ as follows. A similar construction, but in the reverse direction, is used by Bonichon et al. [5].

▶ **Definition 14.** *Let G be a 3-connected planar graph and let S be the minimal Schnyder wood of G^σ. Define the internal triangulation $\tau(G)$ of G and the Schnyder wood of the σ-suspension of $\tau(G)$ to be the graph and Schnyder wood obtained by modifying every internal face f of G as follows (Figure 4). Let P be the counterclockwise 3-colored path on the boundary of f and let $v_1, \dots, v_k$ be its vertices in counterclockwise order around f. If $k \geq 3$, proceed as follows. Add 3-colored edges $v_1v_k, \dots, v_{k-2}v_k$ directed towards v_k and for $j = 2, \dots, k-1$ change the color and orientation of v_jv_{j+1} such that v_jv_{j+1} is 2-colored and directed towards v_j. Proceed the same way for the counterclockwise 1-colored path and the counterclockwise 2-colored path on the boundary of f.*

Observe that if G is k-connected, then so is $\tau(G)$.



(a) An internal face of G. **(b)** The corresponding subgraph of $\tau(G)$.

Figure 4 Illustration for the definition of $\tau(G)$. The counterclockwise 3-colored path P on the boundary of the face of G is highlighted in yellow.

▶ **Lemma 15.** *For a minimal Schnyder wood of G^σ, Definition 14 yields a minimal Schnyder wood of the σ-suspension of $\tau(G)$.*

▶ **Definition 16.** *Let G be a 3-connected plane graph. Let S be a minimal Schnyder wood of G^σ such that r_1r_3 and r_3r_2 are both edges of G. Define the subgraph H of $\tau(G)$ as follows (Figure 5). Let $V(H) = V(\tau(G))$. The edge set of H is defined in the following. The edges on the outer face of $\tau(G)$ that are either 1-2-colored or 2-3-colored are defined to be in $E(H)$. Now, we define which of the internal edges of $\tau(G)$ are in $E(H)$. For this, we treat the 1-2-colored edges and the 1-3-colored edge like unidirected 1-colored edges. The 2-3-colored edge we treat like a unidirected 2-colored edge. Then, for all $i \in \{1, 2, 3\}$, an internal i-colored edge e of $\tau(G)$ is in $E(H)$ if e and two $(i + 1)$-colored edges form a face. Observe that, by Definition 2c, this face needs to be right of e w.r.t. the orientation of e.*

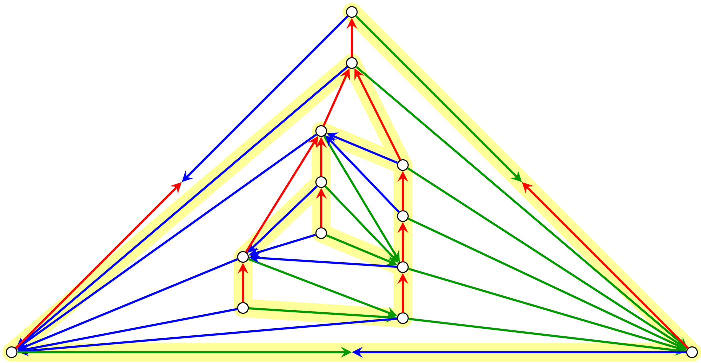


Figure 5 Illustration for the definition of H. H is depicted in yellow.

▶ **Lemma 17.** *As in Definition 16, treat the 1-2-colored edges and the 1-3-colored edge like unidirected 1-colored edges and the 2-3-colored edge like a unidirected 2-colored edge. An edge of $\tau(G)$ with head x is in $E(H)$ if and only if it is the first incoming i-colored edge in clockwise direction at x for some $i \in \{1, 2, 3\}$.*

▶ **Lemma 18.** *H has maximum degree at most 3.*

Sketch of proof. Treat the edges of the outer face of $\tau(G)$ as described in Definition 16. Let $v \in V(H)$ be a vertex that is not a root vertex. It is possible to argue for the root vertices in a similar way. We show that, for every $i \in \{1, 2, 3\}$, of the incoming i-colored and the outgoing $(i + 1)$-colored edges at v there is at most one edge in $E(H)$. Assume that at v there is an incoming edge $e = vy \in E(H)$, and w.l.o.g. e is 2-colored. By Definition 16, e and the outgoing 3-colored edge vx at v form a triangle with another 3-colored edge xy. By Definition 2c, xy is incoming 3-colored at x. Hence, xy precedes vx in clockwise order around x. Thus, by Lemma 17, $vx \notin E(H)$. Furthermore, by Lemma 17, e is the only incoming 2-colored edge at v that is in H.

If at v there is an outgoing edge $vw \in E(H)$, and w.l.o.g. vw is 3-colored. Then, by Definition 16, vw and the at v outgoing 1-colored edge vu are on the boundary of a face together with another 1-colored edge. By Definition 2c, incoming 2-colored edges at v only occur in the clockwise sector between vw and vu. As vw and vu are on the same face, this sector is empty and thus there is no incoming 2-colored edge at v.

Hence, for every $i \in \{1, 2, 3\}$, of the set of the incoming i-colored edges and the outgoing $(i + 1)$-colored edge at v there is at most one edge in $E(H)$. Those three sets cover all edges incident to v and hence $\deg_H(v) \leq 3$. Similar arguments show that also the root vertices have degree at most 3. ◀

▶ **Lemma 19.** *H is a subgraph of G.*

Proof. Let $e \in E(\tau(G)) \setminus E(G)$ be w.l.o.g. 3-colored with tail v and head w. In the following, we show that e is no edge of H. This implies that H is a subgraph of G. Let f be the face of $\tau(G)$ that has v and w on the boundary in that clockwise order. Let $e' = wu$ be the edge succeeding e on f in clockwise order. As observed in Definition 14, wu is incoming 3-colored at w (Figure 4). If $w \neq r_3$, then wv is not the in clockwise order first incoming 3-colored edge at w in the sense of Lemma 17. Hence, $wv \notin E(H)$.

So let $w = r_3$. Assume, for the sake of contradiction, that $u = r_1$. The edge $uv = r_1v$ succeeding $wu = r_3r_1$ on f in clockwise order is outgoing 2-colored at u by Definition 14 (Figure 4). The only outgoing 2-colored edge at r_1 is on the clockwise path from r_1 to r_2 on the outer face of $\tau(G)$. And hence, v is on that path. As vr_3 is unidirected, $v \neq r_2$. Hence, $\{v, r_3\}$ is a 2-separator of $\tau(G)$ and thus of G, contradicting the 3-connectivity of G. This implies that $u \neq r_1$ and thus $r_1r_3 \neq wu$. As above, wv is not the in clockwise order first incoming 3-colored edge at w in the sense of Lemma 17, and thus, $wv \notin E(H)$. ◀

▶ **Lemma 20.** *The co-graph $\neg H^*$ of H in G has maximum degree at most 3. All edges in $E(\neg H^*)$ except $(r_1r_3)^*$ are bidirected.*

Sketch of proof. We show that all bidirected edges of G except r_1r_3 are in H. Let $e = xy$ be a bidirected internal edge in G.

Case 1. x and y are both internal vertices of G. Assume that e is w.l.o.g. a 2-3-colored edge in G. By Definition 14, e becomes 2-colored and is on a face with two 3-colored edges e_1 and e_2 in $\tau(G)$. As x and y are both internal vertices, e_1 and e_2 are both internal edges and, by Lemma 5, unidirected. And thus, $e \in E(H)$ by Definition 16.

Case 2. x and y are both on the outer face of G. If they do not appear consecutively, then they form a 2-separator of G, contradicting the 3-connectivity of G. Thus, xy is an edge on the outer face of G. Then, by Definition 16, $e \in E(H)$ if and only if $e \neq r_1r_3$.

Case 3. W.l.o.g. x is on the outer face of G and y is an internal vertex. This case follows with similar arguments as Case 1.

Hence, only $(r_1r_3)^*$ and the dual edges of unidirected edges might be edges of $\neg H^*$. By Corollary 8, the dual edges of unidirected edges in G are bidirected. Also, observe that $(r_1r_3)^*$ is unidirected and points into the outer face of G. Thus, for an internal face f of G only the outgoing edges of f^* might be in $\neg H^*$. Hence, $\deg_{\neg H^*}(f^*) \leq 3$. As, by Definition 16, only one edge on the boundary of the outer face of G is not in H, the dual of the outer face has degree 1 in $\neg H^*$. ◀

The following definition and lemmas are in preparation of the final statement (Proposition 26) of this section. They study the structure of $\tau(G)$ under the assumption that H is not connected. In the end, they allow us to show that if H is not connected, then there is a 3-separator in G (Figure 6). This yields that if G is 4-connected, then H is connected.

▶ **Definition 21.** *For $x \in V(\tau(G))$, define $DFS(x)$ to be the DFS-index of x for a depth first search on T_1 that starts at r_1 and explores the children of each vertex in counterclockwise order.*

For a vertex $x \in V(\tau(G))$ let $p_3(x)$ and $p_1(x)$ be the parent of x in T_3 and T_1, respectively (Figure 6). Let $c_2(x)$ be the vertex such that $c_2(x)x$ is the clockwise first incoming 2-colored edge at x if existent, i.e., $c_2(x)$ is the clockwise first child of x in T_2. Define $ld(x)$ ("leftmost descendant") to be the descendant v of x in T_1 such that v is a leaf of T_1 and $DFS(v)$ is minimal.

For a set of vertices $S \subseteq V(\tau(G))$ denote the set of the descendants of S in T_1 by $desc(S)$. Define $desc^+(S) := desc(S) \cup S$.

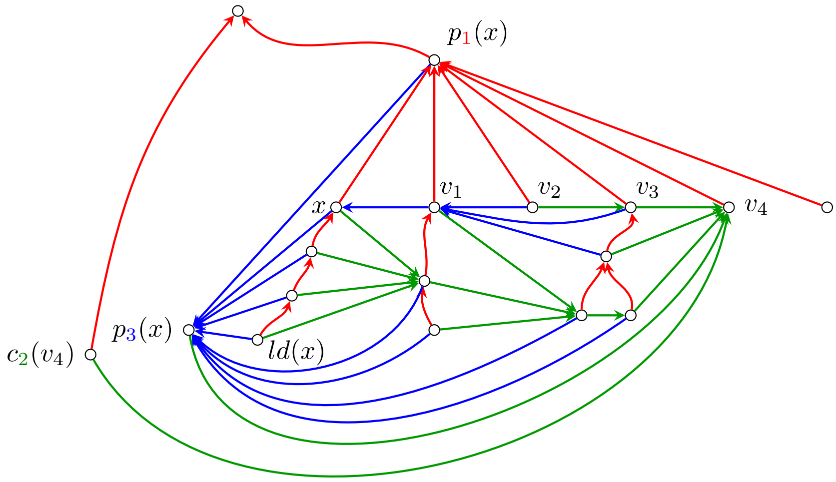


Figure 6 Illustration for Proposition 26. If x is not connected to a vertex with smaller DFS-index in H, then $p_3(x)$, $p_1(x)$ and $v_j = v_4$ form a separating triangle.

▶ Remark 22. Remember that, by Lemma 4, $T_1{}^{-1} \cup T_2 \cup T_3{}^{-1}$ is acyclic. Observe that the DFS-index is a total order of $T_1{}^{-1} \cup T_2 \cup T_3{}^{-1}$. E.g. for an edge xy that is 1-colored incoming at y we have that $DFS(x) > DFS(y)$.

Also, observe that we obtain the path from x to $ld(x)$ in T_1 by descending T_1 starting at x and choosing the counterclockwise first child until we hit a leaf of T_1.

▶ **Lemma 23.** *Assume x is not connected to $p_1(x)$ in H, i.e., there is no path in H that has x and $p_1(x)$ as endpoints. Then, the situation in $\tau(G)$ is as follows. Let $v_0, \ldots, v_k$ be the DFS-ordered children of $p_1(x)$ in T_1, i.e., $DFS(v_0) < \ldots < DFS(v_k)$, and $x = v_t$ for some $t \in \{0, \ldots, k\}$. Then, there exists $j \in \{t+1, \ldots, k\}$ such that $c_2(v_j)$ exists and is not in $desc^+(v_t, \ldots, v_k)$ (Figure 6). We say that x has property B.*

Proof. Let the *height* of a vertex v in T_1 be the length of a longest oriented path in T_1 from a leaf of T_1 to v. The proof is by induction on the height of $p_1(x)$ in T_1. Assume that this height is one. Then, the vertices $x = v_t, \ldots, v_k$ are all leaves in T_1. We now show that if x does not have property B, then x is connected to $p_1(x)$ in H. So assume that for every $i \in \{t+1, \ldots, k\}$ either $c_2(v_i)$ does not exist or $c_2(v_i)$ is a vertex of $v_t, \ldots, v_k$ in T_1.

If $c_2(v_i)$ does not exist, for some $i \in \{t+1, \ldots, k\}$, i.e., v_i does not have an incoming 2-colored edge, then, as $\tau(G)$ is internally 3-connected, $v_i v_{i-1}$ exists and is 3-colored by Definition 2c. As $v_i p_1(x)$ and $v_{i-1} p_1(x)$ are 1-colored, $v_i v_{i-1}$ is in H by Definition 16. Otherwise, $c_2(v_i) = v_j$ for some $j \in \{t, \ldots, k\}$. As observed in Definition 21, $DFS(c_2(v_i)) < DFS(v_i)$ and hence $j \in \{t, \ldots, i-1\}$. Since $c_2(v_i)v_i$ is in $E(H)$ by Lemma 17, v_j and v_i are connected in H. Hence, in any case v_i is connected in H to a vertex v_j with $j \in \{t, \ldots, i-1\}$. This finally yields that $x = v_t$ and v_k are connected in H. As $v_k p_1(x)$ is the clockwise first incoming 1-colored edge at $p_1(x)$, $v_k p_1(x)$ is in $E(H)$ by Lemma 17. And thus, $x = v_t$ and $p_1(x)$ are connected in H.

Assume that the height of $p_1(x)$ is at least two. Again, we show that if x does not have property B, then x and $p_1(x)$ are connected in H. As before, we observe that if $c_2(v_i)$ does not exist for some $i = t+1, \ldots, k$, then v_i and v_{i-1} are connected in H. Also, if $c_2(v_i) = v_j$ for some $j \in \{t, \ldots, k\}$, then $j < i$ and v_j and v_i are connected in H.

So assume that $c_2(v_i)$ is a descendant of v_j, $j \in \{t,\dots,k\}$. For the same reason as above, $j \in \{t,\dots,i-1\}$. Let P be the 1-colored path from $c_2(v_i)$ to v_j in $\tau(G)$. P, $c_2(v_i)v_i$, $v_i p_1(x)$ and $v_j p_1(x)$ form a cycle C. Observe that all vertices of C except for $p_1(x)$ are descendants of $p_1(x)$. Thus, by planarity, for every vertex in the interior of C the outgoing 1-colored path meets $p_1(x)$ or a descendant of $p_1(x)$. And thus, all vertices in the interior of C are descendants of $p_1(v)$ in T_1. Let $y \neq v_j$ be a vertex of P. As $p_1(y)$ is a descendant of $p_1(x)$, it has smaller height in T_1 than $p_1(x)$. We want to apply induction on y. Hence, we need to show that y does not have property B.

Assume, for the sake of contradiction, that y has property B. Then, there is a child z of $p_1(y)$ such that $DFS(y) < DFS(z)$, $c_2(z)$ exists and for every child b of $p_1(y)$ with $DFS(y) \leq DFS(b)$, $c_2(z)$ is neither a descendant nor $c_2(z) = b$. Especially, $DFS(c_2(z)) < DFS(y)$. All vertices of C and all vertices in its interior, except for vertices on the 1-colored path from $p_1(y)$ to $p_1(x)$, have a higher DFS-index than y. By Lemma 4, $c_2(z)$ cannot be on this path from $p_1(y)$ to $p_1(x)$. Thus, $c_2(z)$ cannot occur on C or in its interior. And hence, $c_2(z)$ needs to be in the exterior of C. As all neighbors of y that have a higher DFS-index than y are in the interior of C, we have that z is in the interior of C. Hence, $c_2(z)$ is in the exterior and z is in the interior of C, violating planarity.

This yields that y does not have property B and by induction y and $p_1(y)$ are connected in H. This holds for every vertex on $P \setminus \{v_j\}$. Thus, v_j and $c_2(v_i)$ are connected in H. By Lemma 17, $c_2(v_i)v_i \in E(H)$, and thus v_j and v_i are connected in H. As before, we obtain that x and $p_1(x)$ are connected in H. ◀

▶ **Lemma 24.** *If $x \in V(\tau(G)) \setminus \{r_1\}$ is not connected to a vertex v with $DFS(v) < DFS(x)$ in H, then x is not connected to $p_1(x)$ in H and x is the child of $p_1(x)$ in T_1 with smallest DFS-index. Furthermore, all vertices w on the x-$ld(x)$-path in T_1 are connected to x in H. And we have for all vertices w on the $p_1(x)$-$ld(x)$-path in T_1 that $p_3(w) = p_3(x)$ (Figure 6).*

Proof. Let $x \in V(\tau(G))$ such that x is not connected to a vertex v with $DFS(v) < DFS(x)$. Let $v_0,\dots,v_k$ be the DFS-ordered children of $p_1(x)$ in T_1. First, we show that $x = v_0$. Assume, for the sake of contradiction, that $x = v_j$ with $j \in \{1,\dots,k\}$. Then, xv_{j-1} has either color 2 or 3. If it has color 3, then, by Definition 16, xv_{j-1} is in H and $DFS(v_{j-1}) < DFS(x)$, a contradiction. If xv_{j-1} has color 2, then this edge is incoming 2-colored at x, by Definition 2c. This implies that $c_2(x)$ exists. $DFS(c_2(x)) < DFS(x)$ and, by Lemma 17, $c_2(x)x \in E(H)$, a contradiction. And hence, $x = v_0$.

Assume, for the sake of contradiction, that there exists a vertex w on the x-$ld(x)$-path in T_1 such that w is not connected to x in H. Choose w such that $DFS(w)$ is minimal. Then, $p_1(w)$ is connected to x in H, and hence, w is not connected to $p_1(w)$ in H. Thus, by Lemma 23, w has property B. Let $w = w_0,\dots,w_k$ be the DFS-ordered children of $p_1(w)$ in T_1. Let j be the maximal index such that $c_2(w_j)$ exists and is not a descendant of $p_1(w)$. Since w has property B, j exists. Since j is maximal, w_j does not have property B. Hence, by Lemma 23, it is connected to $p_1(w)$ in H and thus to x. Since $c_2(w_j)w_j \in E(H)$, x is connected to $c_2(w_j)$ in H.

As observed in Definition 21, $DFS(c_2(w_j)) < DFS(w_j)$. The DFS-indices of the vertices $w_0,\dots,w_{j-1}$ and their descendants are exactly the indices in between $DFS(w_j)$ and $DFS(w_0)$. And the DFS-indices of the vertices on the $p_1(w)$-x-path are exactly the indices in between $DFS(p_1(w))$ and $DFS(x)$. By Lemma 4, $c_2(w_j)$ cannot be on the $p_1(w)$-x-path. Hence, $DFS(c_2(w_j)) < DFS(x)$, a contradiction. Hence, w is connected to x in H.

As all vertices on the x-$ld(x)$-path in T_1 are connected to x in H, they all do not have incoming 2-colored edges. Indeed, incoming 2-colored edges at those vertices would connect x to a vertex with smaller DFS-index. Since every vertex on the x-$ld(x)$-path needs to form a triangle with its parent, the vertex and the parent send their 3-colored outgoing edge to the same vertex $p_3(x)$. ◀

Similar arguments are used to show the following lemma.

▶ **Lemma 25.** *Let $x \in V(\tau(G)) \setminus \{r_1\}$ be not connected to a vertex v with $DFS(v) < DFS(x)$ in H. Let $x = v_0, \dots, v_k$ be the DFS-ordered children of $p_1(x)$ in T_1. Let v_j be the vertex of smallest index such that $c_2(v_j)$ exists and is not in $desc^+(v_0, \dots, v_{j-1})$. Then, the edge $v_j p_3(x)$ is in $E(\tau(G))$.*

▶ **Proposition 26.** *If G is 4-connected, then H is connected.*

Proof. Observe that if G is 4-connected, then so is $\tau(G)$. We show that every vertex $x \neq r_1$ is connected in H to a vertex with lower DFS-index. Assume, for the sake of contradiction, that x is not connected to a vertex with lower DFS-index. With the help of the previous lemmas we show that $p_3(x)$, $p_1(x)$ and v_j (as defined in Lemma 23) form a 3-separator in H. Let $v_0, \dots, v_k$ be the DFS-ordered children of $p_1(x)$ in T_1.

By Lemma 23, there exists a vertex v_j, $j \in \{1, \dots, k\}$ of smallest DFS-index such that $c_2(v_j)$ exists and is not in $desc^+(v_0, \dots, v_{j-1})$. Observe that $p_1(x)v_j$ is an edge of $\tau(G)$.

By Lemma 24, $x = v_0$ and all vertices on the $p_1(x)$-$ld(x)$-path in T_1 send their 3-colored outgoing edge to the same vertex $p_3(x)$. Thus, $p_1(x)p_3(x) \in E(\tau(G))$.

By Lemma 25, the edge $v_j p_3(x)$ exists in $\tau(G)$. Hence, the edges $v_j p_3(x)$, $p_1(x)p_3(x)$ and $p_1(x)v_j$ form a triangle in $\tau(G)$. The vertex x is in the interior of this triangle. Assume, for the sake of contradiction, that r_1 is not in the exterior of this triangle. Then, $r_1 = p_1(x)$ and, by Lemma 24, $x = r_3$. By Definition 16, r_3 is connected to r_1 by the clockwise path on the outer face from r_1 to r_3. As $DFS(r_1) < DFS(r_3)$, we arrive at a contradiction. Hence, r_1 is in the exterior of the triangle of $v_j p_3(x)$, $p_1(x)p_3(x)$, i.e., this triangle is a separating triangle in $\tau(G)$, contradicting the 4-connectivity of $\tau(G)$. This yields that every vertex, except for r_1, is connected to a vertex with lower DFS-index in H, and thus H is connected. ◀

4 A Tree of Maximum Degree 3 and a Co-Tree of Maximum Degree 4

In this section, we give one lemma on the structure of ordered path partitions. Then, we finally prove the main theorem.

We want to remind the reader of the definition of ordered path partitions and the fact that the maximal 2-3-colored paths of a Schnyder wood yield the compatible ordered path partition $\mathcal{P}^{2,3}$.

▶ **Lemma 27** ([20]). *Let G be a 4-connected plane graph, S be the minimal Schnyder wood of G^σ and $\mathcal{P}^{2,3} = (P_0, \dots, P_s)$ be the ordered path partition that is compatible with S. Let $P_i := (v_1, \dots, v_k) \neq P_0$ be a path of $\mathcal{P}^{2,3}$ and v_0 and v_{k+1} be its left and right neighbor. Then, every edge $v_l w \notin \{v_0v_1, v_kv_{k+1}\}$ with $v_l \in P_i$ and $w \in V_{i-1}$ is unidirected, 1-colored and incoming at v_k and $w \notin \{v_0, v_{k+1}\}$.*

▶ **Theorem 28.** *Every 4-connected planar graph G contains a 3-tree such that its co-tree is a 4-tree.*

Sketch of proof. Let S be a minimal Schnyder wood of G such that r_1r_3 and r_3r_2 are both edges of G. By Lemma 9, the completion $\widetilde{G}_S$ of G contains no clockwise directed cycle. By

Proposition 26 and Lemma 19, G has a connected subgraph H as defined in Definition 16. It has maximum degree at most 3 by Lemma 18, and its co-graph has maximum degree at most 3 by Lemma 20. Now, we define a subset D of the edges of H such that $H - D$ becomes acyclic and the degree of its co-graph does not exceed 4.

Let C be a cycle in H. Let $\mathcal{P}^{2,3} = (P_0, \ldots, P_s)$ be the compatible ordered path partition of S. Let P be the path of maximal length in C such that $P \subseteq P_M = (v_1, \ldots, v_k)$ with $M := \max\{i \mid P_i \cap V(C) \neq \emptyset\}$. P is the *index maximal* subpath of C.

Since $P \subseteq P_M$ is the index maximal subpath of the cycle C, there need to be two edges in H that join a vertex of P with a vertex of V_{M-1}. Say that those edges are the *associated* edges of P. Remember that $V_i := \bigcup_{q=0}^{i} V(P_q)$. The Schnyder wood S is minimal. Hence, by Lemma 27, every edge except for v_0v_1 and v_kv_{k+1} that joins a vertex of P_M with a vertex of V_{M-1} is unidirected, 1-colored and incoming in v_k. Let e_P be the clockwise first incoming 1-colored edge at v_k if existent. Otherwise, let e_P be v_kv_{k+1}. It is possible to show that v_0v_1 and e_P are the only edges in H that join a vertex of P with a vertex of V_{M-1}. This also directly yields that $P = P_M$.

Let $\mathcal{P}_{max}$ be the set of all index maximal subpaths of cycles in H. Now, we need to identify for each $P = (v_1, \ldots, v_k) \in \mathcal{P}_{max}$ an edge of the set $E(P) \cup \{v_0v_1, e_P\}$ such that removing those edges leaves H acyclic and connected and does not raise the maximum degree of $\neg H^*$ above 4. Define $D = D_{uni} \cup D_{bi}$ to be this set of edges. Start with $D_{uni} = D_{bi} = \emptyset$.

If e_P is unidirected, add it to D_{uni}. Otherwise, if e_P is bidirected and v_0v_1 is unidirected add v_0v_1 to D_{uni}. By Lemma 20, all edges of $\neg H^*$ except for $(r_1r_3)^*$ are bidirected. As all edges in D_{uni} are unidirected, their duals are bidirected, by Corollary 8. Hence, all edges in $\neg H^* + D^*_{uni}$ except for $(r_1r_3)^*$ are bidirected. By the same reasoning as in the proof of Lemma 20, the maximum degree of $\neg H^* + D^*_{uni}$ is at most 3.

Let $\mathcal{P}^{bi}_{max} \subseteq \mathcal{P}_{max}$ be the paths $P = (v_1, \ldots, v_k)$ such that both v_0v_1 and e_P are bidirected. Observe that in this case $e_P = v_kv_{k+1}$. Hence, we now are in the same situation as in the proof of the main theorem of [20]. We are able to apply the exact same arguments in order to obtain the desired set D_{bi} such that $H - D$ becomes acyclic and $\neg H^* + D^*$ remains acyclic and has maximum degree at most 4. Since H has maximum degree at most 3, so does $H - D$. And thus, $H - D$ and $\neg H^* + D^*$ are the desired trees. ◀

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