# Schnyder woods and long induced paths in 3-connected planar graphs ⋆

Christian Ortlieb

Institute of Computer Science, University of Rostock, Germany

**Abstract** In the recent 30 years, Schnyder woods have become an invaluable asset in the study of planar graphs. We contribute to this research with a brief and comprehensible proof of a new structural feature: every Schnyder wood of a 3-connected planar graph on $n$ vertices has a tree of depth at least $\lfloor 1/6 \log_2 n \rfloor$. As a simple implication, our result improves the previous hard-won lower bound on the length of an induced path in such a graph to $1/6 \log_2 n$.

**Keywords:** Induced paths · 3-connected planar graphs · Schnyder woods · depth

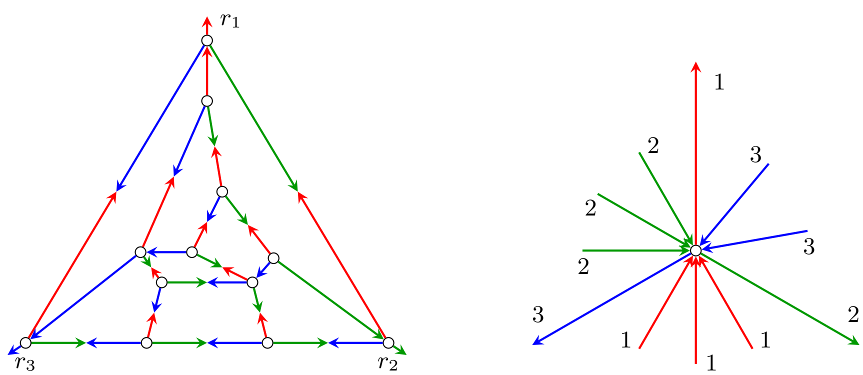
## 1 Introduction

The problem of finding a long induced path has first been investigated by Erdős et al. [5] already in 1986. Let $p(G)$ be the size, i.e. the number of vertices, of a longest induced path of $G$. Erdős et al. [5] gave the lower bound $p(G) \geq 2r(G) - 1$ for a connected graph $G$ in terms of the radius $r(G)$ of $G$. In 2000, Arocha and Valencia [3] found a lower bound of $\log_\Delta(n)$ on the diameter (and hence on $p(G)$) of a 3-connected planar graph $G$ with bounded maximum degree $\Delta$, and the number of vertices $n$. If $\Delta$ is not bounded, they show that there is an induced path of size $\sqrt{\log_3(\Delta)}$. In 2016, Di Giacomo et al. [4] gave the lower bound $p(G) \geq \frac{\log_2 n}{12 \log_2 \log_2 n}$ for 3-connected planar graphs $G$. They also showed that there exist 3-connected planar graphs $G$ with $p(G) \leq 1.3 \log_2(n) + 5$. Also in 2016, Esperet et al. [6] could improve the lower bound of Di Giacomo et al. [4] to $1/6(\log_2(n) - 3 \log_2 \log_2(n))$ choosing a similar approach.

Here, we show the slightly better lower bound $p(G) \geq 1/6 \log_2(n)$. But, our proof is by far simpler than the previous ones of Di Giacomo et al. [4] and Esperet et al. [6] and also uses a completely different approach which yields new structural insights. These new insights are discussed at the end of the introduction. The next paragraph gives the reader an intuition about Schnyder woods.

⋆ This research is supported by the grant SCHM 3186/2-1 (401348462) from the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation). This preprint has not undergone peer review or any post-submission improvements or corrections. The Version of Record of this contribution is published in LNCS, and is available online at https://doi.org/10.1007/978-3-031-55601-2_2.

*The concept of Schnyder woods* is widely used in graph drawing and related areas [1,2,8,7,11]. A Schnyder wood of a 3-connected planar graph is a triple of directed spanning trees such that every edge is in at least one and at most two trees. The three trees are colored red, blue and green. In every tree, the edges are oriented towards the root. If an edge is in two different trees it has opposite orientations in the two trees. Consider for example the red edges in the graph in Figure 1a. They form a spanning tree rooted at $r_1$. Additionally, around each vertex the outgoing and ingoing edges need to obey a certain pattern. In the clockwise sector between the outgoing red and the outgoing green edge only ingoing blue edges occur. For the ingoing red and green edges symmetric properties hold. See Figure 1b for illustration. A formal definition is given in Section 2. In this paper we exploit the fact that a path from a leaf to the root in a tree of a Schnyder wood needs to be an induced path. See Corollary 1 for this argument. So in order to find a long induced path, we investigate the depth, i.e. the length of a longest path from a leaf to the root, of all three trees in a Schnyder wood of a given 3-connected planar graph.




(a) A Schnyder wood of the suspension of a 3-connected planar graph.

(b) Example for Condition 3 at a vertex in a Schnyder wood. The ingoing edges in color $i$ are in the clockwise sector between the outgoing edge in color $i+1$ and the outgoing edge in color $i-1$.

Figure 1: Illustrations for the definition of Schnyder woods.

Given a planar embedding of a 3-connected planar graph and a Schnyder wood on this embedding, we show that at least one of the three trees has depth at least $1/6\log_2(n)$, which, as mentioned above, directly implies that $p(G) \geq 1/6\log_2(n)$. To the best of our knowledge, we are the first to investigate the depth of the trees of Schnyder woods. This new structural property of Schnyder woods is not only of theoretical interest, but also yields the following extra information on the problem of finding long induced paths.

We easily obtain a linear time approach computing an induced path of the desired size. In fact, given a 3-connected, planar graph, we compute a Schnyder wood in linear time [7,10,11]. Then, we compute a longest leaf-to-root path for each of the three trees of the Schnyder wood in linear time by traversing the three trees e.g. with a breadth-first search. The longest such path has size at least $1/6\log_2(n)$.

Furthermore, we know that for every choice of three vertices which are on the boundary of a common face, we can find an induced path of size at least $1/6\log_2(n)$ that ends in one of the three vertices. It is easy to derive that there are at least $f/(2\Delta)$ different such paths, where $f$ is the number of faces and $\Delta$ the maximum degree.

Finally, for each such path there exists a grid drawing such that the path is monotone in both coordinates. Such a drawing can be found using for example the algorithm of Felsner [7] which is based on Schnyder woods and yields a drawing on the $(f-1)\times(f-1)$ grid.

To conclude the paper, we show that, for every $n \geq 3$, there exists a 3-connected planar graph with a Schnyder wood such that the three trees have depth at most $\log_3(2n-5)+1 \approx 0.63\cdot\log_2(n)+2$ using the approach of Di Giacomo et al. [4].

## 2 Preliminaries

We use standard graph notation. The graphs $G$ we consider in this paper are simple, planar, 3-connected and come with a fixed embedding into the plane, that is, $G$ is plane.

We use the definition of Schnyder woods as given by Felsner [8]. The *suspension* $G^\sigma$ of $G$ is obtained by choosing three different vertices $r_1$, $r_2$ and $r_3$ which appear in clockwise order on the outer face and by adding adjacent to each of those vertices a half-edge which reaches into the outer face. With a little abuse of notation, we define a *half-edge* as an arc starting at a vertex but with no defined end vertex. The special vertices $r_1$, $r_2$ and $r_3$ are called *roots*.

Given a suspension $G^\sigma$, a *Schnyder wood* rooted at $r_1$, $r_2$ and $r_3$ is an orientation and coloring of the edges with colors 1, 2 and 3 satisfying the following conditions. Indices indicating colors are modulo 3. This means that e.g. $3+1\equiv 1$.

1. Every edge is either oriented in one direction (unidirected edge) or in both directions (bidirected edge). Every edge receives a distinct color for every direction. So unidirected edges receive one color and bidirected edges two different colors.
2. For every $i \in \{1,2,3\}$ the half-edge at $r_i$ is directed away from $r_i$ and colored $i$.
3. For every vertex $v$ and every color $i \in \{1,2,3\}$ there is exactly one incident outgoing (half-)edge of color $i$. The outgoing edges $e_1$, $e_2$ and $e_3$ of $v$ in colors 1, 2 and 3, respectively, occur clockwise around $v$. The ingoing edges of $v$ in color $i$ are in the clockwise sector from $e_{i+1}$ to $e_{i-1}$.

4. No interior face has a boundary which is a directed cycle in one color.

See Figure 1b and 1a for illustration. For ease of notation we define a Schnyder wood of $G$ to be a Schnyder wood of a suspension of $G$. Throughout the paper we use red, green and blue synonymously for color 1, 2 and 3, respectively. Denote by $T_i$ the directed graph induced by the (uni- and bidirected) edges that have color $i$. $T_1$, $T_2$ and $T_3$ are called the *trees* of the Schnyder wood.

The following properties of Schnyder woods are used in this paper.

**Lemma 1 (Felsner [7]).** *Every 3-connected plane graph has a Schnyder wood.*

**Lemma 2 (Felsner [7]).** *$T_i$ is a directed tree rooted at $r_i$ for every $i \in \{1,2,3\}$.*

For a directed graph $H$, let $H^{-1}$ be the graph obtained from $H$ by reversing the orientation of all edges.

**Lemma 3 (Felsner [7]).** *For all $i \in \{1,2,3\}$ $T_i \cup T_{i-1}^{-1} \cup T_{i+1}^{-1}$ does not have an oriented cycle.*

For every vertex $v \in V(G)$ denote by $P_i(v)$ the path from $v$ to the root in the tree $T_i$. The *depth* of $T_i$ is the length (number of edges) of a longest path from a vertex to the root in $T_i$. For ease of notation, $P_i(v)$ also denotes the vertex set of the path $P_i(v)$. Hence, $|P_i(v)|$ is the number of vertices of $P_i(v)$ and $P_j(w) \cap P_i(v)$ is the set of vertices $P_j(w)$ and $P_i(v)$ have in common.

Denote by $R_i(v)$ the region bounded by and including $P_{i-1}(v)$, $P_{i+1}(v)$ and the clockwise path from $r_{i+1}$ to $r_{i-1}$ on the outer face. See Figure 2 for an illustration. For example, vertices on the path $P_{i-1}(v)$ belong to both $R_i(v)$ and $R_{i+1}(v)$.

**Lemma 4 (Felsner [8]).** *Let $i \in \{1,2,3\}$. If $u \in R_i(v)$ then $R_i(u) \subseteq R_i(v)$.*

Similar observations as in Felsner's proof of Lemma 4 directly yield the following lemma.

**Lemma 5.** *Let $i \in \{1,2,3\}$. If $u \in R_i(v) \setminus P_{i+1}(v)$, then $R_i(u) \setminus P_{i+1}(u) \subsetneq R_i(v) \setminus P_{i+1}(v)$ and $v \notin R_i(u) \setminus P_{i+1}(u)$.*

*Proof.* We prove the claim for $i = 1$. For $i \in \{2,3\}$, the proof is symmetric. Let $u \in R_1(v) \setminus P_2(v)$. By Lemma 4, $R_1(u) \subseteq R_1(v)$. We now need to prove that $P_2(v)$ intersects $R_1(u)$ only in $P_2(u)$. By definition, $R_1(u)$ is bounded by $P_2(u)$, $P_3(u)$ and the clockwise path $P$ from $r_2$ to $r_3$ on the outer face. Since $P_2(v)$ is on the boundary of $R_1(v)$ and $R_1(u) \subseteq R_1(v)$, we know that $P_2(v)$ can intersect $R_1(u)$ only on its boundary, meaning $P_2(v) \cap R_1(u) \subseteq P_3(u) \cup P_2(u) \cup P$.

In the following, we show that $P_2(v) \cap (P \setminus P_2(u)) = \emptyset$. Let $x_v$ be the first vertex of $P_2(v)$, starting at $v$, which is also on $P$ and let $x_u$ be the first vertex of $P_2(u)$, starting at $u$, which is also on $P$. Assume, for the sake of contradiction, that $x_v$ comes after $x_u$ on $P$ starting at $r_2$. Since $u \in R_1(v)$, there needs to be a vertex $y$ at which $P_2(u)$ leaves $R_1(v)$. But $P_2(u) \subseteq R_1(u) \subseteq R_1(v)$, so we arrive at a contradiction. So $x_u$ comes after $x_v$ on $P$ starting at $r_2$ and $P_2(v)$ does not intersect $P \setminus P_2(u)$. Thus $P_2(v) \cap R_1(u) \subseteq P_3(u) \cup P_2(u)$.

For the sake of contradiction, assume that $P_2(v) \cap P_3(u)$ is non-empty. Let $x$ be the first vertex on the path $P_3(u)$ starting at $u$ which is also in $P_2(v)$. If $x = u$, then $u \in P_2(v)$, contradicting the assumption of the lemma. So $x \neq u$. Since $u \in R_1(v)$, there is either an outgoing 2-colored, an ingoing 3-colored and an outgoing 3-colored edge around $x$ in that clockwise order or there is an outgoing 3-colored, an ingoing 3-colored and an ingoing 2-colored edge around $x$ in that clockwise order. This both contradicts Condition 3 at $x$.

Remember that $P_2(v) \cap R_1(u) \subseteq P_3(u) \cup P_2(u)$. Additionally, we now have that $P_2(v) \cap P_3(u) = \emptyset$, which then yields $P_2(v) \cap R_1(u) \subseteq P_2(u)$. Together with the above mentioned fact that $R_1(u) \subseteq R_1(v)$, this yields that $R_1(u) \setminus P_2(u) \subseteq R_1(v) \setminus P_2(v)$. Since $u \in R_1(v) \setminus P_2(v)$ but $u \notin R_1(u) \setminus P_2(u)$, we have $R_1(u) \setminus P_2(u) \subsetneq R_1(v) \setminus P_2(v)$. As $v \in P_2(v)$ and $P_2(v) \cap R_1(u) \subseteq P_2(u)$, we obtain $v \notin R_1(u) \setminus P_2(u)$. □

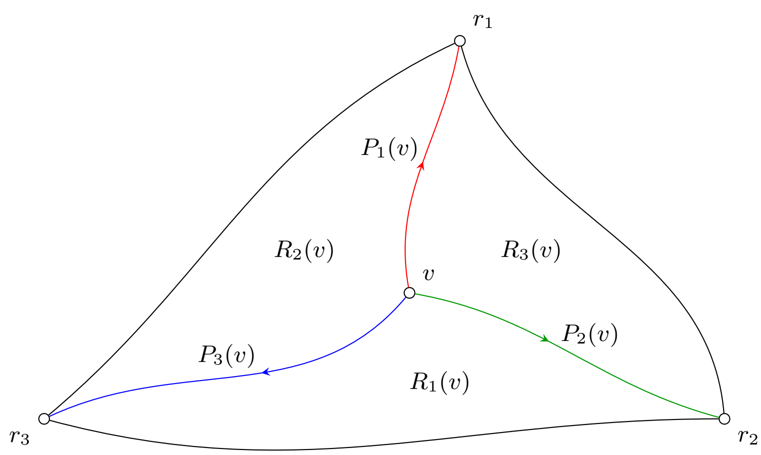



Figure 2: Illustration of the definition of regions and paths for a vertex $v$.

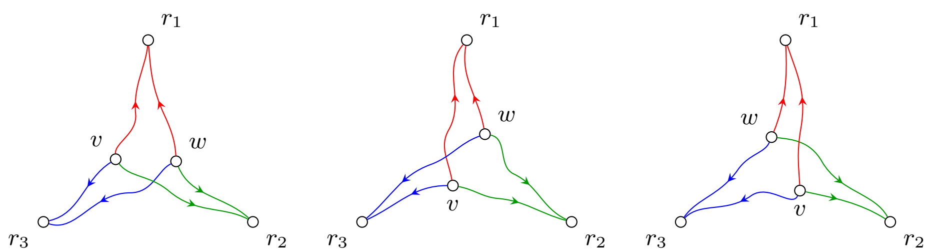
## 3 Lower Bound on the Maximum Depth of a Tree

In this section we bound the maximum depth of a tree in a Schnyder wood of a 3-connected planar graph $G$ on $n$ vertices from below by $\lfloor 1/6 \log_2(n) \rfloor$. As a corollary we derive that $G$ has an induced path of size at least $\lfloor 1/6 \log_2(n) \rfloor + 1$.

**Theorem 1.** *Let $G$ be a 3-connected plane graph and let $T_1$, $T_2$ and $T_3$ be the trees of a Schnyder wood of $G$. Then at least one, $T_1$, $T_2$ or $T_3$, has depth at least $\lfloor 1/6 \log_2(n) \rfloor$.*

The general idea of the proof is the following. For any set $C \subseteq V(G)$ and $v \in C$, we consider the set of vertices which are on $P_1(v)$ and also on a path $P_3(w)$ for some $w \in C$. Naturally, this set $\bigcup_{w \in C}(P_1(v) \cap P_3(w))$ is a subset of $P_1(v)$. The size of this set gives a natural lower bound on the depth on $T_1$ and is later denoted as $l_1^C(v)$.

Now, we need to find a set of vertices $C$ and a vertex $x \in C$ such that $\bigcup_{w \in C}(P_1(x) \cap P_3(w))$ (or $\bigcup_{w \in C}(P_3(x) \cap P_1(w))$) is large enough, i.e. has size at least $\lfloor 1/6 \log_2(n) \rfloor + 1$. Let $C$ be a set such that for two vertices $v, w \in C$, $v \neq w$, we have that either $v \in R_1(w) \setminus P_2(w)$ and $w \in R_3(v) \setminus P_1(v)$ or vice versa $v \in R_3(w) \setminus P_1(w)$ and $w \in R_1(v) \setminus P_2(v)$. See Figure 3b for illustration. We show that such a set $C$ of size at least $n^{1/3}$ exists, and that there is a vertex $x \in C$ such that either $\bigcup_{w \in C}(P_1(x) \cap P_3(w))$ or $\bigcup_{w \in C}(P_3(x) \cap P_1(w))$ has size at least $\lfloor 1/2 \log_2(|C|) \rfloor + 1$, as required. Together this then yields the desired lower bound of $\lfloor 1/6 \log_2(n) \rfloor$ on the depth of either $T_1$ or $T_3$, respectively.




(a) Two vertices in $A$. (b) Two vertices in $C$. $v \leq_2 w$ in $\mathcal{P}'$. (c) Two vertices in $A'$.

Figure 3: Relation between vertices in the sets $A$, $C$ and $A'$. For every vertex $x \in \{v, w\}$ the paths $P_i(x)$, $i = 1, 2, 3$ are drawn.

*Proof.* First, we show that a set $C$ with properties symmetric to the above properties exists. Those properties are: $|C| \geq n^{1/3}$ and for some $i \in \{1, 2, 3\}$ we have for every $v, w \in C$, $v \neq w$, that either $v \in R_i(w) \setminus P_{i+1}(w)$ and $w \in R_{i+2}(v) \setminus P_i(v)$ or vice versa $v \in R_{i+2}(w) \setminus P_i(w)$ and $w \in R_i(v) \setminus P_{i+1}(v)$ .

We define the following relation $\leq_1$ on $V(G)$: For $v, w \in V(G)$ let $v \leq_1 w$ if $v \in R_1(w) \setminus P_2(w)$ or $v = w$. The relation $\leq_1$ is obviously reflexive. By Lemma 5, $\leq_1$ is antisymmetric, since $w \notin R_1(v) \setminus P_2(v)$ for $v \leq_1 w$ and $v \neq w$. Also by Lemma 5, for $u \leq_1 v$, $v \leq_1 w$ with $v \neq w$, we have either $u \in R_1(v) \setminus P_2(v) \subseteq R_1(w) \setminus P_2(w)$ or $u = v$. Therefore $u \leq_1 w$. Hence $\leq_1$ is transitive and, thus, $\mathcal{P} := (V(G), \leq_1)$ is a poset.

By Mirsky's theorem [9], we either find a chain $L$ of size at least $n^{2/3}$ or we can decompose $\mathcal{P}$ into at most $n^{2/3}$ antichains. In the latter case, we find, by the pigeonhole principle, an antichain $A$ of size $n/n^{2/3} = n^{1/3}$. Consider the set $A$. For any two vertices $v, w \in A$, $v \neq w$, we have that $v \notin R_1(w) \setminus P_2(w)$ and

$w \notin R_1(v) \setminus P_2(v)$. Thus $v$ is either in $R_3(w) \setminus P_1(w)$ or $R_2(w) \setminus P_3(w)$ . Assume w.l.o.g. that the latter applies. Then $w \notin R_2(v) \setminus P_3(v)$, by Lemma 5. And thus $w$ needs to be in $R_3(v) \setminus P_1(v)$. This is symmetric to the property which we are aiming for and thus $A$ is already fine for us. See Figure 3a for illustration.

Otherwise, if the chain $L$ of $\mathcal{P}$ exists, we define the relation $\leq_2$ on $L$: For $v, w \in L$, let $v \leq_2 w$ if $w \in R_3(v) \setminus P_1(v)$ or $v = w$. As above, $\mathcal{P}' := (L, \leq_2)$ is a poset and we either find a chain $C$ of size $n^{1/3}$ or an antichain $A'$ of size $n^{2/3}/n^{1/3} = n^{1/3}$. For two vertices $v, w \in C$, $v \neq w$, we have w.l.o.g. $v \leq_2 w$ and hence $w \in R_3(v) \setminus P_1(v)$. Since $v$ and $w$ are in $L$, a chain in $\mathcal{P}$, we either have $w \leq_1 v$ or $v \leq_1 w$. If $w \leq_1 v$ then $w \in R_1(v) \setminus P_2(v)$, contradicting $w \in R_3(v) \setminus P_1(v)$. So $v \leq_1 w$ and hence $v \in R_1(w) \setminus P_2(w)$.

Similarly we obtain for two vertices $v, w \in A'$, $v \neq w$, that $v \in R_1(w) \setminus P_2(w)$ and $w \in R_2(v) \setminus P_3(v)$ or vice versa. Again, see Figure 3. Since the relations on $A$, $A'$ and $C$ are symmetric, we assume w.l.o.g. that $C$ exists.

Let $S \subseteq C$. For all $v \in S$ and all $(i, j) \in \{(1, 3), (3, 1)\}$ define

$$l_i^S(v) := |\bigcup_{x \in S} (P_i(v) \cap P_j(x))|,$$

that is, the number of vertices on $P_i(v)$ that are also on a path $P_j(x)$ for some $x \in S$. For example, $l_1^S(v)$ is the number of vertices on $P_1(v)$ that are also on a path $P_3(x)$ for any $x \in S$.

Let, furthermore,

$$l_i^S := \max_{v \in S} l_i^S(v).$$

and let $\omega_i^S$ be a vertex of $S$, that realizes this maximum value, i.e. $l_i^S(\omega_i^S) = l_i^S$.

In the following, we prove by induction that $l_3^C + l_1^C \geq \lfloor \log_2(|C|) \rfloor + 2$. Our bound then follows by pigeonhole principle. Clearly, if $|C| = 1$, we have $l_3^C + l_1^C = 2 = \log_2(1) + 2$ and the claim holds.

Let $|C| \in \{2, 3\}$. Then we have two vertices $v, w \in C$, $v \neq w$, such that $v \leq_2 w$ in $\mathcal{P}'$. So $v \in R_1(w) \setminus P_2(w)$ and $w \in R_3(v) \setminus P_1(v)$. Especially $w \notin P_1(v)$ and hence $P_1(v)$ intersects $P_3(w)$ in a vertex different from $w$. See Figure 3b for illustration. This yields $l_3^C(w) \geq 2$. Thus

$$l_3^C + l_1^C \geq l_3^C(w) + l_1^C(v) \geq 2 + 1 = 3 = \lfloor \log_2(3) \rfloor + 2 \geq \lfloor \log_2(|C|) \rfloor + 2.$$

So assume that $|C| \geq 4$. Partition $C = C_1 \cup C_2 \cup X$ such that $|C_1| = |C_2| = 2^z$, $z$ is maximal, and $C_1 = \{v_1, \dots, v_s\}$, $C_2 = \{v_{s+1}, \dots, v_{2s}\}$ with $v_1 \leq_2 \dots \leq_2 v_{2s}$ in $\mathcal{P}'$. By induction, $l_3^{C_k} + l_1^{C_k} \geq \lfloor \log_2(|C_k|) \rfloor + 2 = \log_2(|C_k|) + 2$ for $k = 1, 2$.

So $v_l \in R_1(\omega_3^{C_2}) \setminus P_2(\omega_3^{C_2})$ and $\omega_3^{C_2} \in R_3(v_l) \setminus P_1(v_l)$ for $l = 1, \dots, s$. Hence $P_1(v_l)$ intersects $P_3(\omega_3^{C_2})$ for $l = 1, \dots, s$. Observe that, by Condition 3, this intersection is a set of vertices that appears consecutively on both $P_1(v_l)$ and $P_3(\omega_3^{C_2})$. And, by Condition 1 and 3, the first vertex of $P_1(v_l)$, starting at $v_l$, that is also on $P_3(\omega_3^{C_2})$ is the vertex where $P_3(\omega_3^{C_2})$ leaves $R_3(v_l)$.

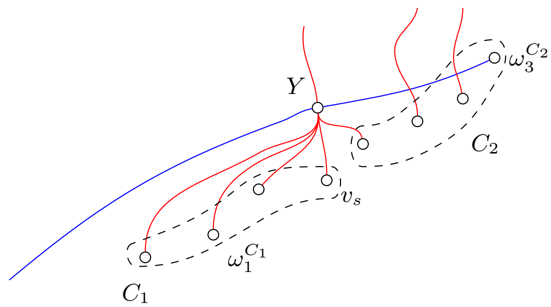



Figure 4: Situation as in Case 1. The blue path from $\omega_3^{C_2}$ to the root intersects the red path from $\omega_1^{C_1}$ to the root.

Define $Y$ to be the intersection of the 1-colored path starting at $v_s$ and the 3-colored path starting at $\omega_3^{C_2}$, i.e. $Y := P_1(v_s) \cap P_3(\omega_3^{C_2})$. For $l = 1, \ldots, s-1$ we have that $v_s \in R_3(v_l) \setminus P_1(v_l) \subseteq R_3(v_l)$. Hence, by Lemma 4, $P_1(v_s) \subseteq R_3(v_s) \subseteq R_3(v_l)$. Let $p \in P_1(v_l)$ and $q \in P_1(v_s)$ be the first vertex, starting at $v_l$ and $v_s$, of $P_1(v_l)$ and $P_1(v_s)$, respectively, that is also on $P_3(\omega_3^{C_2})$. As observed above, $P_3(\omega_3^{C_2})$ leaves $R_3(v_s)$ at $q$ and $R_3(v_l)$ at $p$. And since $R_3(v_s) \subseteq R_3(v_l)$ the vertex $q$ occurs before $p$ on $P_3(\omega_3^{C_2})$, starting at $\omega_3^{C_2}$. Since $T_1$ is a tree, we know that if $P_1(v_l) \cap P_1(v_s) \cap P_3(\omega_3^{C_2}) \neq \emptyset$, then the first vertex of $P_3(\omega_3^{C_2})$, starting at $\omega_3^{C_2}$, that is also on $P_1(v_s)$ coincides with the first vertex of $P_3(\omega_3^{C_2})$, starting at $\omega_3^{C_2}$, that is also on $P_1(v_l)$. So if $P_1(v_l) \cap P_3(\omega_3^{C_2}) \setminus Y = \emptyset$, then $p = q$ and $P_1(v_l) \cap P_3(\omega_3^{C_2}) = Y$. Thus the following case distinction is exhaustive. Either all 1-colored paths starting at vertices of $C_1$ intersect $P_3(\omega_3^{C_2})$ in the set $Y$ or there is a vertex in $C_1$ such that its 1-colored path intersects $P_3(\omega_3^{C_2})$ in a vertex not in $Y$. We distinguish those two cases.

**Case 1:** Assume that $P_1(v_l) \cap P_3(\omega_3^{C_2}) = Y$ for $l = 1, \ldots, s-1$, see Figure 4. So especially $P_1(\omega_1^{C_1}) \cap P_3(\omega_3^{C_2}) = Y$. We now show that no 3-colored path starting at a vertex in $C_1$ intersects $Y$. So assume for the sake of contradiction that there is an $l \in \{1, \ldots, s\}$ such that $P_3(v_l) \cap Y \neq \emptyset$.

If $v_l \notin Y$ then $P_3(v_l)$ intersects $P_1(v_l)$ in a vertex different from $v_l$. This directly yields a directed cycle in $T_1 \cup T_2^{-1} \cup T_3^{-1}$, contradicting Lemma 3. So, assume $v_l \in Y$. Then $v_l \in P_1(v_t)$ for some $t \in \{1, \ldots, s\} \setminus \{l\}$, contradicting the definition of $C$.

So $P_3(v_l) \cap Y = \emptyset$ for $l = 1, \ldots, s$. Remember that $Y$ equals the intersection of $P_1(\omega_1^{C_1})$ and $P_3(\omega_3^{C_2})$. So we have that

$$\begin{aligned} l_1^C(\omega_1^{C_1}) &= |\bigcup_{x \in C} (P_1(\omega_1^{C_1}) \cap P_3(x))| \\ &\geq |\bigcup_{x \in C_1} (P_1(\omega_1^{C_1}) \cap P_3(x))| + |Y| = l_1^{C_1}(\omega_1^{C_1}) + |Y| \\ &\geq l_1^{C_1}(\omega_1^{C_1}) + 1 = l_1^{C_1} + 1, \end{aligned}$$

and we obtain

$$\begin{aligned} l_3^C + l_1^C &\geq l_3^{C_1} + l_1^C(\omega_1^{C_1}) \geq l_3^{C_1} + l_1^{C_1} + 1 \\ &\geq \log_2(|C_1|) + 2 + 1 = \log_2(2|C_1|) + 2 \\ &= \lfloor \log_2(|C|) \rfloor + 2. \end{aligned}$$

**Case 2:** Assume that there is a vertex $v_l$, $l \in \{1, \ldots, s-1\}$ such that $Z := (P_1(v_l) \cap P_3(\omega_3^{C_2})) \setminus Y \neq \emptyset$, see Figure 5. We now show that no 1-colored path starting at a vertex in $C_2$ intersects $Z$. Assume for the sake of contradiction that there is a vertex $v_t$, $t \in \{s+1, \ldots, 2s\}$ such that $P_1(v_t) \cap Z \neq \emptyset$.

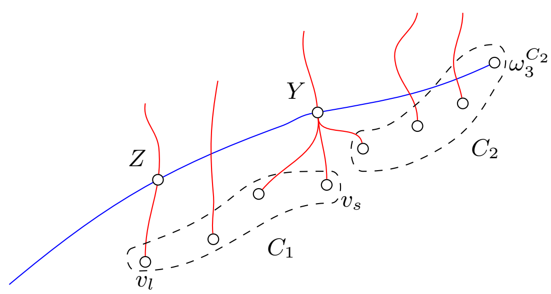



Figure 5: Situation as in Case 2. The red path from $v_l$ to the root intersects the blue path from $\omega_3^{C_2}$ to the root.

In the following we show that $Z \cap R_3(v_s) = \emptyset$. Then, since $v_t \in R_3(v_s) \setminus P_1(v_s)$, the 1-colored path $P_1(v_t)$ starting at $v_t$ starts in $R_3(v_s)$ and leaves it at some vertex. This will lead to a contradiction.

We have that $v_l \leq_2 v_s$ in $\mathcal{P}'$ and $v_l \neq v_s$ so $v_s \in R_3(v_l) \setminus P_1(v_l)$. By Lemma 5, $R_3(v_s) \setminus P_1(v_s) \subsetneq R_3(v_l) \setminus P_1(v_l)$. Assume there exists a vertex $z \in Z \cap R_3(v_s)$. Either $z \in P_1(v_s)$, contradicting the definition of $Z$, or $z \in R_3(v_s) \setminus P_1(v_s)$. Then we have $z \notin R_3(v_l) \setminus P_1(v_l) \supsetneq R_3(v_s) \setminus P_1(v_s) \ni z$, a contradiction. So $Z \cap R_3(v_s) = \emptyset$. As mentioned above $v_t \in R_3(v_s) \setminus P_1(v_s)$ since $v_s \leq_2 v_t$ and $v_s \neq v_t$. This situation requires the 1-colored path $P_1(v_t)$ starting at $v_t$ to leave $R_3(v_s)$ at some vertex $w \in R_3(v_s)$.

If this vertex $w \in P_1(v_s)$, it would have two outgoing 1-colored edges. And if $w \in P_2(v_s) \setminus \{v_s\}$, it would have an outgoing 2-colored edge, an outgoing 1-colored edge and an ingoing 2-colored edge in that clockwise order. In both cases Condition 3 would be violated at $w$.

So, $P_1(v_t) \cap Z = \emptyset$ for $t = s+1, \ldots, 2s$. And as in Case 1 we have $l_3^C(\omega_3^{C_2}) \geq l_3^{C_2}(\omega_3^{C_2}) + |Z| \geq l_3^{C_2}(\omega_3^{C_2}) + 1 = l_3^{C_2} + 1$. And we obtain that

$$\begin{aligned} l_3^C + l_1^C &\geq l_3^C(\omega_3^{C_2}) + l_1^{C_2} \geq l_3^{C_2} + 1 + l_1^{C_2} \\ &\geq \log_2(|C_2|) + 2 + 1 = \log_2(2|C_2|) + 2 \\ &= \lfloor \log_2(|C|) \rfloor + 2. \end{aligned}$$

By pigeonhole principle, there is an $i \in \{1,3\}$ such that $l_i^C \geq \lceil 1/2 \lfloor \log_2(|C|) \rfloor + 1 \rceil \geq \lfloor 1/2 \log_2(n^{1/3}) \rfloor + 1 = \lfloor 1/6 \log_2(n) \rfloor + 1$. As $l_i^C - 1$ is a lower bound on the depth of $T_i$ the claim follows. □

In 2016, Esperet et al. [6] showed that in a 3-connected planar graph there exists an induced path with at least $1/2(1/3 \log_2(n) - \log_2 \log_2(n))$ vertices. Using Theorem 1, we can do slightly better.

**Corollary 1.** *Every 3-connected planar graph $G$ on $n$ vertices has an induced path of size $\lfloor 1/6 \log_2(n) \rfloor + 1$.*

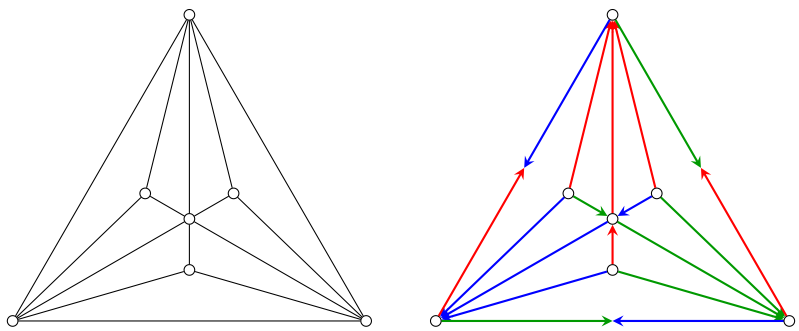
*Proof.* For a vertex $v \in V(G)$ the paths $P_1(v)$, $P_2(v)$ and $P_3(v)$ are always induced. Assume that there exists a vertex $v \in V(G)$ for which this does not hold. Then there is an $i$-colored edge $e = xy$ in $G$ with $x, y \in P_j(v)$, $i, j \in \{1,2,3\}$. By Lemma 2, $T_i$ is a tree and hence $i \neq j$. Now, either $T_i^{-1} \cup T_j$ or $T_i \cup T_j$ has an oriented cycle, contradicting Lemma 3. So, Theorem 1 directly yields an induced path of size $\lfloor 1/6 \log_2(n) \rfloor + 1$. □

## 4 On the Tightness of the Lower Bound

The natural question is, if there are graphs $G_n$ of size $n$ with Schnyder woods such that the maximum depth of a tree in the Schnyder wood is bounded from above by a function of $n$.

Di Giacomo et al. [4] considered this kind of question for induced paths. They showed that for every $n \geq 3$ there exists a 3-connected planar graph $G$ on $n$ vertices such that the longest induced path in $G$ has size at most $2\log_3(2n-5) + 3 \approx 1.3 \cdot \log_2(n) + 5$ using planar 3-trees. We use the same graphs to show that there are 3-connected planar graphs $G_n$ of size $n$ with Schnyder woods such that the maximum depth of any of its trees is at most $\log_3(2n-5) + 1 \approx 0.63 \cdot \log_2(n) + 2$.

We use the definition of *(almost) complete planar 3-trees* as given by Di Giacomo et al. [4]: $G_0$ is a triangle and is defined to be a complete planar 3-tree. We obtain $G_{i+1}$ from $G_i$ by placing a vertex into each internal face and connecting this vertex to the vertices on the face boundary. Concerning the Schnyder wood of those graphs, we observe: $G_0$ has a unique Schnyder wood $S_0$. We obtain $S_{i+1}$ from $S_i$ by the following operation: The edges already in $G_i$ retain their orientation and coloring. We assign the only possible orientation and coloring to the new edges which does not violate Condition 3. See Figure 6 for illustration. An *almost complete planar 3-tree* $\hat{G}_i$ is a graph which is constructed as follows. We take a complete planar 3-tree $G_{i-1}$ and only add in a subset of the internal faces new vertices. They are connected as above and also the Schnyder wood of $\hat{G}_i$ is obtained as above. Complete planar 3-trees are also almost complete planar 3-trees. Now, take an almost complete planar 3-tree $\hat{G}_i$ and, for every vertex $v \notin V(G_0)$, define the *level* of $v$ to be the smallest integer $i$ such that $v$ is in $\hat{G}_i$ but not in $G_{i-1}$. For vertices of $G_0$ define the level to be 0.



(a) The complete planar 3-tree $G_2$. (b) $G_2$ together with its Schnyder wood.

Figure 6: Illustration for the definition of complete planar 3-trees.

**Lemma 6.** *For every $n \geq 3$ there exists a 3-connected planar graph with a Schnyder wood such that the trees $T_1$, $T_2$ and $T_3$ all have depth at most* $\log_3(2n-5)+1$.

*Proof.* Let $\hat{G}_i$ be an almost complete planar 3-tree with $n$ vertices. $G_{i-1}$ has $\frac{3^{i-1}+5}{2}$ vertices and hence $n \geq \frac{3^{i-1}+5}{2}$. Every leaf of a tree of the Schnyder wood of $\hat{G}_i$ has level at most $i$ and the path from a vertex to the root in a tree of the Schnyder wood is strictly decreasing in level. So the maximum depth of a tree in $\hat{G}_i$ is at most $i \leq \log_3(2n-5)+1$. $\square$

## 5 Conclusion

We showed that every Schnyder wood of a 3-connected planar graph $G$ on $n$ vertices has a tree of depth at least $1/6\log_2(n)$. As a leaf-to-root path in a tree of a Schnyder wood is an induced path in $G$, this yields an induced path of size at least $1/6\log_2(n)$. Schnyder woods are well investigated objects. So without any additional effort we obtain a linear time algorithm which finds such an induced path of size at least $1/6\log_2(n)$. Also, there is a grid drawing such that the long induced path is monotone in both coordinates and we know that there are at least $f/(2\Delta)$ different such paths. Here $f$ is the number of faces of $G$ and $\Delta$ its maximum degree. Furthermore, every improvement of the bound on the depth of trees in Schnyder woods directly leads to the improvement of the bound on the length of induced paths. We are confident, that via this approach further improvements are possible.

We also showed that there exists a graph with a Schnyder wood such that each of the three trees has depth at most $\log_3(2n-5)+1 \approx 0.63 \cdot \log_2(n)+2$. This gives an idea where the limits of this new method might be.

**Acknowledgment** The author would like to express his gratitude to Christian Rosenke for his valuable input.

## References

1. Aerts, N., Felsner, S.: Straight-line triangle representations via Schnyder labelings. J. Graph Algorithms Appl. **19**(1), 467–505 (2015). https://doi.org/10.7155/jgaa.00372
2. Alam, J., Evans, W., Kobourov, S., Pupyrev, S., Toeniskoetter, J., Ueckerdt, T.: Contact representations of graphs in 3d. In: Dehne, F., Sack, J.R., Stege, U. (eds.) Algorithms and Data Structures. pp. 14–27. Springer International Publishing, Cham (2015). https://doi.org/10.1007/978-3-319-21840-3_2
3. Arocha, J.L., Valencia, P.: Long induced paths in 3-connected planar graphs. Discuss. Math. Graph Theory **20**(1), 105–107 (2000). https://doi.org/10.7151/dmgt.1110
4. Di Giacomo, E., Liotta, G., Mchedlidze, T.: Lower and upper bounds for long induced paths in 3-connected planar graphs. Theoret. Comput. Sci. **636**, 47–55 (2016). https://doi.org/10.1016/j.tcs.2016.04.034
5. Erdős, P., Saks, M., Sós, V.T.: Maximum induced trees in graphs. J. Combin. Theory Ser. B **41**(1), 61–79 (1986). https://doi.org/10.1016/0097-3165(86)90115-9
6. Esperet, L., Lemoine, L., Maffray, F.: Long induced paths in graphs. European J. Combin. **62**, 1–14 (2017). https://doi.org/10.1016/j.ejc.2016.11.011
7. Felsner, S.: Convex drawings of planar graphs and the order dimension of 3-polytopes. Order **18**(1), 19–37 (2001). https://doi.org/10.1023/A:1010604726900
8. Felsner, S.: Geometric graphs and arrangements. Advanced Lectures in Mathematics, Friedr. Vieweg & Sohn, Wiesbaden (2004). https://doi.org/10.1007/978-3-322-80303-0
9. Mirsky, L.: A dual of Dilworth's decomposition theorem. Amer. Math. Monthly **78**, 876–877 (1971). https://doi.org/10.2307/2316481
10. Miura, K., Azuma, M., Nishizeki, T.: Canonical decomposition, realizer, Schnyder labeling and orderly spanning trees of plane graphs. Internat. J. Found. Comput. Sci. **16**(1), 117–141 (2005). https://doi.org/10.1142/S0129054105002905
11. Schnyder, W.: Embedding planar graphs on the grid. In: Proceedings of the first annual ACM-SIAM symposium on Discrete algorithms. pp. 138–148 (1990)