

Counting K_4 -Subdivisions

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Abstract

A fundamental theorem in graph theory states that any 3-connected graph contains a subdivision of K_4 . As a generalization, we ask for the minimum number of K_4 -subdivisions that are contained in every 3-connected graph on n vertices. We prove that there are $\Omega(n^3)$ such K_4 -subdivisions and show that the order of this bound is tight for infinitely many graphs. We further investigate a better bound in dependence on m and prove that the computational complexity of the problem of counting the exact number of K_4 -subdivisions is $\#P$ -hard.

1 Introduction

Subdivisions of the complete graph K_4 on four vertices play a prominent role in graph structure theory: They do not only form the inductive anchor for constructive characterizations of 3-connectivity such as Tutte's Wheel Theorem [6] or Barnette and Grünbaum's characterization [1], they also received considerable attention in the variants of K_4^- -subdivisions (where K_4^- is a K_4 minus one edge) and *totally odd K_4 -subdivisions* (in which every subdivided edge is of odd length) due to applications for colorings, planarity and parity constrained disjoint paths problems [5, 3].

It is folklore that every 3-connected graph contains a K_4 -subdivision (see [1] for an early reference); with containment we always mean the usual subgraph-relation. In terms of connectivity, this is optimal, since 2-connected graphs do not necessarily contain a K_4 -subdivision, as cycles or the arbitrarily large graphs $K_{2,n-2}$ show. In terms of numbers, it is optimal, as the minimal 3-connected graph K_4 contains exactly one K_4 -subdivision.

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As a generalization, we ask for the minimum number $\phi(n)$ of pairwise different K_4 -subdivisions that are contained in every 3-connected graph on n vertices. The dependence on n will allow to prove more than just one such subdivision. We will prove that $\phi(n) \in \Omega(n^3)$ and that this lower bound is tight up to constant factors. We also show that there is a better lower bound when the input graph has many edges, namely $\Omega(m^4/n)$. Finally, we show that the computational problem of counting these K_4 -subdivisions exactly is $\#P$ -hard. Clearly, the *maximal* number of different K_4 -subdivisions may be exponential in n , as the complete graphs show.

2 Preliminaries

We will consider only finite and simple graphs. A *subdivision* of a graph H (a *H-subdivision*) is a graph obtained from H by replacing every edge with a path of length at least one. A vertex of a H -subdivision is called *real* if it has degree at least three in H and *unreal* otherwise.

A *k-separator* of a graph $G = (V, E)$ is a set of k vertices whose deletion leaves a disconnected graph. Let $n := |V|$ and $m := |E|$. A graph G is *k-connected* if $n > k$ and G contains no $(k - 1)$ -separator. A path from a vertex s to a vertex t is called an *s-t-path* (and contains every vertex at most once). A set of paths is called *independent* if the intersection of the vertices of every two paths is a subset of $\{s, t\}$.

We first give an upper bound for the minimal number of K_4 -subdivisions in 3-connected graphs with n vertices. Consider a wheel-graph (see Figure 1). Every K_4 -subdivision of such a graph contains the central vertex as real vertex, as otherwise there are at most two real vertices instead of the desired four. The remaining part of the subdivision is then uniquely defined by choosing 3 real vertices arbitrarily on the rim of the wheel. This implies the upper bound $\phi(n) \leq \binom{n-1}{3}$ and thus $\phi(n) \in O(n^3)$.

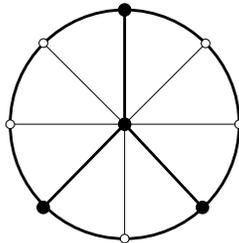


Figure 1: A K_4 -subdivision of a wheel-graph (fat edges). The black vertices are real vertices.

For an adequate lower bound of the same order, we will first show some useful facts about the minimum number of cycles in 2-connected graphs.

3 Cycles in 2-Connected Graphs

An *open ear decomposition* of a 2-connected graph $G = (V, E)$ is a sequence (P_1, P_2, \dots, P_l) of subgraphs of G partitioning E such that P_1 is a cycle and every $P_i \neq P_1$ is a path that intersects $P_1 \cup \dots \cup P_{i-1}$ in exactly its endpoints. Each P_i is called an (open) *ear* [4, 8].

Open ear decompositions are known to exist for and only for 2-connected graphs. For each i , $P_1 \cup \dots \cup P_i$ is again 2-connected. We will first establish a lower bound on the minimum number of cycles in 2-connected graphs, which is dependent on the number of ears. The first lemma ensures that there are many distinct paths with fixed endvertices.

Lemma 1. *Let s and t be two vertices in a 2-connected graph G with l ears. Then G contains $l + 1$ distinct s - t -paths.*

Proof. The proof proceeds by induction on the number of ears in an open ear decomposition of G . If $l = 1$, G is a cycle and the claim follows. If $l > 1$, let G' be the 2-connected graph $P_1 \cup \dots \cup P_{l-1}$. By induction hypothesis, G' contains l distinct s - t -paths for any two vertices s and t . Let a and b be the two end vertices of P_l . We distinguish three cases (see Figure 2) and prove for each case that G contains an additional s - t -path.

1. $s \in V(G')$ and $t \in V(G')$:

It suffices to show that there is an s - t -path in G that contains P_l ; this path differs from the other l paths. Consider the graph H that is obtained from G by adding a new vertex v with neighbors s and t and by subdividing an edge of P_l with the vertex w . As H is 2-connected, there is a cycle in H containing v and w by Menger's Theorem, which gives the desired s - t -path containing P_l in G .

2. $s \in V(G')$ and $t \notin V(G')$ (or, by symmetry, vice versa):

Then t is an inner vertex of P_l . By induction, we have l distinct s - a -paths in G' . Extending each of these paths to t along P_l gives l distinct s - t -paths in G . An additional s - t -path can be obtained by extending an s - b -path to t along P_l .

3. $s \notin V(G')$ and $t \notin V(G')$:

There are l a - b -paths in G' , each of which can be extended to s - t -paths in G . An additional s - t -path in G is the one in P_l . \square

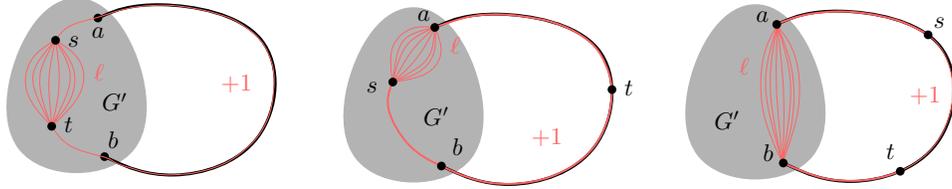


Figure 2

Lemma 1 is used to prove the following lower bound on the number of cycles.

Lemma 2. *Every 2-connected graph G with l ears contains $\binom{l+1}{2}$ distinct cycles.*

Proof. By induction on l . If $l = 1$, G is a cycle and the claim follows. If $l > 1$, let G' be the 2-connected graph $P_1 \cup \dots \cup P_{l-1}$. Then G' has $l - 1$ ears and contains $\binom{l}{2}$ distinct cycles by induction hypothesis. That are l cycles less than we need to show for G . We prove that there are l cycles in G , each of which contains P_l , which gives the claim. Let a and b be the end vertices of P_l . According to Lemma 1, there are l distinct a - b -paths in G' . Augmenting each of these paths with P_l gives the desired l additional cycles. \square

Whitney proved that every open ear decomposition has exactly $m - n + 1$ ears [8]. The number $m - n + 1$ can be easily obtained by deleting one arbitrary edge from each ear, as the resulting graph will be a tree satisfying $m = n - 1$. Applying the number of ears to Lemma 2 gives immediately the following corollary.

Corollary 3. Every 2-connected graph contains $\binom{m-n+2}{2}$ distinct cycles.

The bound of Corollary 3 is tight (for all n and $m = 2n - 4$), as the graphs $K_{2,n-2}$ show. If additionally the minimum degree in G is δ , we have $m \geq \delta n / 2$ and get the following result.

Corollary 4. Every 2-connected graph with minimum degree δ contains $\binom{n(\delta/2-1)+2}{2} = (\delta - 2)^2 n^2 / 8 + 3(\delta - 2)n / 4 + 1$ distinct cycles.

4 Counting K_4 -Subdivisions

For a vertex v in a 3-connected graph G , let G_v be the graph obtained from G by deleting v . Let d_1, \dots, d_n be the vertex degrees of G (by 3-connectivity,

these are at least three) and, for a vertex v , let d_v be the degree of v in G . Instead of counting K_4 -subdivisions directly in G , we will count cycles in the different graphs G_v and augment these cycles to K_4 -subdivisions using the following corollary of Menger's theorem.

Lemma 5 (Fan Lemma [2, Proposition 9.5]). *Let v be a vertex in a k -connected graph G and let C be a set of at least k vertices in G with $v \notin C$. Then there are k independent paths P_1, \dots, P_k from v to distinct vertices $c_1, \dots, c_k \in C$ such that $V(P_i) \cap C = c_i$ for each $1 \leq i \leq k$.*

Every cycle C in G_v gives a K_4 -subdivision of G by applying the Fan Lemma with v , C and $k = 3$. Every K_4 -subdivision can occur from at most 4 graphs G_v , as v has to be a real vertex of that K_4 -subdivision. Thus, each K_4 -subdivision is counted at most 4 times. We will show that the numbers of cycles for every G_v , $v \in V(G)$, sum up to a large value, namely to the value $c \in \Omega(n^3)$. This implies the desired lower bound $\frac{c}{4} \in \Omega(n^3)$ for the number of K_4 -subdivisions.

It remains to show that $c \in \Omega(n^3)$. Note that each G_v is 2-connected, as it only differs from G by the deletion of one vertex. Moreover, each G_v has exactly $m - d_v$ edges and $n - 1$ vertices. According to Corollary 3, G_v contains at least $\binom{m - d_v - n + 3}{2} =: \binom{a - d_v + 1}{2}$ cycles, where we define $a := m - n + 2$ for brevity. Note that $a - d_v + 1 \geq 2$ is positive, since $m - d_v \geq n - 1$ because G_v is 2-connected. We calculate the total number of cycles c in all G_v as follows.

$$\begin{aligned}
c &\geq \sum_{i=1}^n \binom{a - d_i + 1}{2} \\
&= \sum_{i=1}^n \frac{a^2 - 2ad_i + d_i^2 + a - d_i}{2} \\
&\geq \frac{1}{2}na^2 + \frac{1}{2}na - m - a \sum_{i=1}^n d_i + \frac{1}{2} \sum_{i=1}^n d_i^2 && \text{(as } \sum_{i=1}^n d_i = 2m) \\
&\geq \frac{1}{2}na^2 + \frac{1}{2}na - m - 2ma + 2m^2/n && (\star, \text{ Cauchy-Schwarz}) \\
&= \frac{1}{2n}(m(n-2) - n^2 + 2n)^2 + \frac{1}{2}na - m \\
&\geq \frac{1}{2n} \left(\frac{3}{2}n^2 - 3n - n^2 + 2n \right)^2 + \frac{1}{2}na - m && \text{(as } m \geq \frac{3}{2}n) \\
&= \frac{1}{8}n^3 - \frac{1}{2}n^2 + \frac{3}{2}n + \frac{1}{2}m(n-2) - \frac{1}{2}n^2
\end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{8}n^3 - \frac{1}{2}n^2 + \frac{3}{2}n + \frac{3}{4}n^2 - \frac{3}{2}n - \frac{1}{2}n^2 && \text{(as } m \geq \frac{3}{2}n\text{)} \\
&= \frac{1}{8}n^3 - \frac{1}{4}n^2
\end{aligned}$$

For \star , we used that $\sum_{i=1}^n d_i^2 \geq \frac{(\sum_{i=1}^n d_i)^2}{n} = \frac{4m^2}{n}$, as $\sum_{i=1}^n d_i \leq \sqrt{n} \sqrt{\sum_{i=1}^n d_i^2}$ follows directly from applying the Cauchy-Schwarz inequality to the all 1- and the degree-vector. Using the upper bound of Section 2, we obtain the following theorem.

Theorem 6. *For every n , $\frac{1}{32}n^3 - \frac{1}{16}n^2 \leq \phi(n) \leq \frac{1}{6}n^3 - n^2 + \frac{11}{6}n + 1 = \binom{n-1}{3}$. Thus, $\phi(n) \in \Theta(n^3)$.*

We conjecture that the upper bound coming from the wheel graphs is actually the right bound.

Conjecture 7. For every n , $\phi(n) = \binom{n-1}{3}$.

5 A Better Bound for Large m

When m is large, we can obtain better lower bounds in dependence on m . Let $\phi(n, m)$ be the minimum number of pairwise different K_4 -subdivisions that are contained in every 3-connected graph having n vertices and m edges. We use the same idea as above, but additionally construct many K_4 -subdivisions from one cycle C in G_v whenever d_v is large.

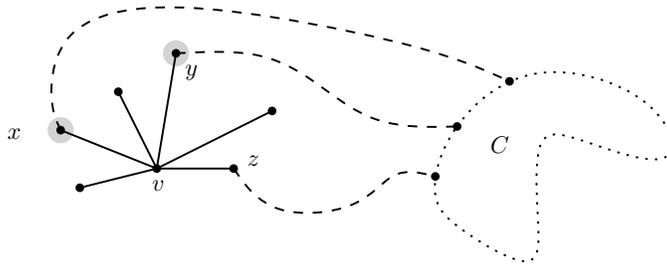


Figure 3

Let $D := \{v, x, y\}$, where x and y are arbitrary distinct neighbors of v in G (D may intersect C). By Menger's theorem, there are three independent C - D -paths in G , possibly of length 0, which we extend to three independent C - v -paths by adding the edges xv and yv . This forces two of the three

independent paths to go through x and y (see Figure 3). The third neighbor z of v in the independent paths cannot be forced this way. However, we may obtain the same K_4 -subdivision containing v, x, y, z and C for G_v only when D is either $\{v, x, y\}$, $\{v, x, z\}$ or $\{v, y, z\}$. Thus, such a K_4 -subdivision is counted at most three times. Since we can choose any two neighbors x and y of v , this gives at least $\lceil \frac{1}{3} \binom{d_v}{2} \rceil$ K_4 -subdivisions for every cycle C in G_v . Thus, we obtain

$$\phi(n, m) \geq \frac{1}{4} \sum_{i=1}^n \left\lceil \frac{1}{3} \binom{d_i}{2} \right\rceil \binom{a - d_i + 1}{2} \quad (\star)$$

We will use this inequality for the following theorem.

Theorem 8. $\phi(n, m) \in \Omega(m^4/n)$.

Proof. If $m < 3n$, $m = \Theta(n)$ and the bound follows directly from $\phi(n) \in \Theta(n^3)$ of Theorem 6. Thus, let $m > 3n$; if we can prove the same asymptotic bound for this case, taking the lower constant factor of the two cases yields the claim. We have $a - d_i + 1 \geq m - 2n + 3 > \frac{m}{3}$ in inequality (\star) . Thus,

$$\begin{aligned} \phi(n, m) &\geq \frac{1}{12} \sum_{i=1}^n \binom{d_i}{2} \binom{\frac{m}{3}}{2} = \frac{1}{24} \binom{\frac{m}{3}}{2} \left(\sum_{i=1}^n d_i^2 - \sum_{i=1}^n d_i \right) \\ &\geq \frac{1}{24} \binom{\frac{m}{3}}{2} \left(4 \frac{m^2}{n} - 2m \right) \quad (\text{Cauchy-Schwarz, } \sum_{i=1}^n d_i = 2m) \\ &= \frac{m}{6} \binom{\frac{m}{3}}{2} \left(\frac{m}{n} - \frac{1}{2} \right) \\ &\in \Omega(m^4/n) \end{aligned}$$

□

6 #P-Hardness

Instead of only giving a lower bound for the number of K_4 -subdivisions, one may try to compute their exact number. Let #SUBDIVISIONS be the problem of counting the exact number $\#K_4(G)$ of K_4 -subdivisions in general graphs. We reduce the following #P-hard problem #S-T-PATHS [7] to the problem #FIXED-SUBDIVISIONS and then to #SUBDIVISIONS, which proves that it is #P-hard.

Problem: **#S-T-PATHS** (this problem is #P-hard [7])

Input: $G; s, t \in V$

Output: Number of different s - t -paths in G .

Problem: **#FIXED-SUBDIVISIONS**

Input: $G; a, b, c, d \in V$

Output: Number of different K_4 -subdivisions in G having a, b, c, d as real vertices.

Theorem 9. *Counting K_4 -subdivisions in general graphs is #P-hard.*

Proof. We first reduce **#S-T-PATHS** to **#FIXED-SUBDIVISIONS**. Given an input (G, s, t) of the first problem, construct the input (G', a, b, c, d) for the second problem such that G' is obtained from a K_4 with vertices $\{a, b, c, d\}$ by replacing the edge ab with the graph $as \cup G \cup tb$ (see Figure 4). Thus, G contains an s - t -path if and only if G' contains an a - b -path not intersecting $\{c, d\}$. It follows that the number of s - t -paths in G is exactly the number of K_4 -subdivisions having real vertices $\{a, b, c, d\}$ in G' .

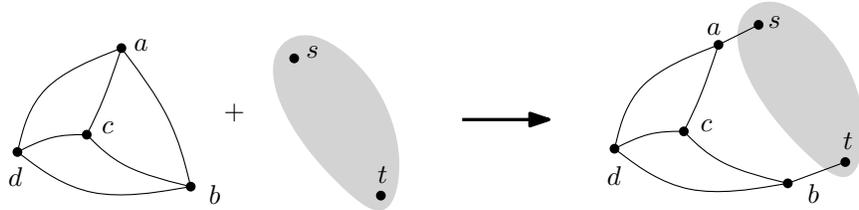


Figure 4

We now reduce **#FIXED-SUBDIVISIONS** to **#SUBDIVISIONS**. Suppose (G', a, b, c, d) is an instance of the first problem; we construct the instance G'' of the second problem by replacing certain edges of G' with the gadget shown in Figure 5. The number of cycles in this gadget is fixed to $s := n^2$, so that 2^s exceeds the maximal number $\#K_4(G')$ of K_4 -subdivisions in G' (which is at most $2^{\binom{n}{2}}$). Clearly, the sizes of G' and G'' are polynomial in the size of G .

To construct G'' from G' , we replace every edge having exactly one endvertex in $\{a, b, c, d\}$ by one gadget and every edge having both endvertices in $\{a, b, c, d\}$ by two gadgets joined in series. These gadgets allow thus 2^s and 2^{2s} different paths between their endvertices, respectively. For convenience,

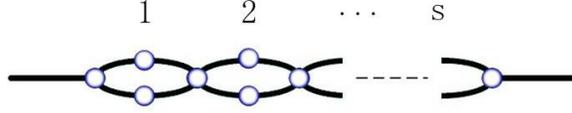


Figure 5

we may see G'' as weighted graph G' for which each edge is weighted with either 1, 2^s or 2^{2s} .

Clearly, no inner vertex of a gadget may be a real vertex of a K_4 -subdivision. Thus, there is the identity mapping between the K_4 -subdivisions in G' and the (weighted) ones in G'' . Suppose there are $N_{x,y}$ K_4 -subdivisions in G' with exactly x real vertices in $\{a, b, c, d\}$ and exactly y unreal vertices in $\{a, b, c, d\}$. Each of them corresponds to a weighted K_4 -subdivision of G'' , for which the product of edge weights is exactly $2^{s(3x+2y)}$. Hence, it corresponds to exactly $2^{s(3x+2y)}$ different K_4 -subdivisions in G'' . In fact, $\#K_4(G'') = \sum_{x+y \leq 4} 2^{s(3x+2y)} N_{x,y}$. As 2^s exceeds $N_{x,y}$, $\lfloor \frac{\#K_4(G'')}{2^{12s}} \rfloor = N_{4,0}$ is the answer for instance (G', a, b, c, d) . \square

While giving $\#P$ -hardness, the above reductions only argue about general graphs. Using the result above, we show the stronger statement that counting K_4 -subdivisions in k -connected graphs is still $\#P$ -hard for every fixed k .

Theorem 10. *For any fixed k , counting K_4 -subdivisions in k -connected graphs is $\#P$ -hard.*

Proof. We can assume $k > 1$, as the arguments in the proof of Theorem 9 hold also for connected graphs. Let G be an instance of $\#SUBDIVISIONS$ and let $\{v_1, \dots, v_n\}$ its vertex set. For a reduction to the problem in question, we construct instances G_s from G by adding $s > n$ new vertices $\{x_1, \dots, x_s\}$ and all edges $x_i v_j$ for $1 \leq i \leq s$ and $1 \leq j \leq n$. Clearly, G_s is k -connected and $n \geq 3$, as $n > k > 1$.

Consider a K_4 -subdivision of G_s . It contains at most 4 real x_i -vertices and at most n v_j -vertices. Thus, it contains at most $3n/2$ unreal x_i -vertices (this is not the best possible bound). In total, it contains at most $3n$ of the x_i -vertices, since $3n/2 \geq 4$.

In every K_4 -subdivision of G_s , we delete all x_i -vertices and call the remaining graph a *partial* K_4 -subdivision. Let N_t , $0 \leq t \leq 3n$, be the number of different partial K_4 -subdivisions of G_s that were generated by deleting exactly t vertices. For the desired k and every integer $r \geq 0$, the number

N_t of G_s with $s := (k + 3n) + r$ is the same, by interchangeability of the x_i -vertices.

For the same reason, each partial K_4 -subdivision that was counted for N_t can be extended to a K_4 -subdivision of G_s in a number $P_{s,t}$ of different ways that is only dependent on s and t : Namely, $P_{s,t} = s!/(s-t)! = s(s-1)\cdots(s-t+1)$, which is the number of ways we can choose t ordered non-repetitive elements from $\{x_1, \dots, x_s\}$. Hence,

$$\#K_4(G_s) = \sum_{t=0}^{3n} P_{s,t}N_t = N_0 + \sum_{t=1}^{3n} s(s-1)\cdots(s-t+1)N_t = N_0 + \sum_{t=1}^{3n} s^t N'_t,$$

where N'_t is some fixed linear combination of $N_t, N_{t+1}, \dots, N_{3n}$ that is not dependent on s .

We construct the graph G_s for each $s \in \{k+3n, k+3n+1, \dots, k+6n+1\}$ and obtain $3n+1$ linear equations in $N_0, N'_1, \dots, N'_{3n}$ (we set $N'_0 := N_0$), whose coefficient matrix M is Vandermonde in s and nonsingular, as all the values of s are pairwise distinct. We thus have the equation $\overrightarrow{\#K_4(G_s)} = M\overrightarrow{N'_t}$ for the corresponding vectors. As M is nonsingular, we can invert it and obtain $\overrightarrow{N'_t} = M^{-1}\overrightarrow{\#K_4(G_s)}$. As we know the elements of $\#K_4(G_s)$ for all s , we get $\overrightarrow{N'_t}$ and therefore in particular N_0 , which is equal to $\#K_4(G)$. \square

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