Shortness parameters of polyhedral graphs with few distinct vertex degrees

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Abstract

We devise several new upper bounds for shortness parameters of regular polyhedra and of the polyhedra that have two vertex degrees, and relate these to each other. Grünbaum and Walther showed that quartic polyhedra have shortness exponent at most $\log 22/\log 23$. This was subsequently improved by Harant to $\log 16/\log 17$, which holds even when all faces are either triangles or of length k, for infinitely many k. We complement Harant's result by strengthening the Grünbaum-Walther bound to $\log 4/\log 5$, and showing that this bound even holds for the family of quartic polyhedra with faces of length at most 7. Furthermore, we prove that for every $4 \leq \ell \leq 8$ the shortness exponent of the polyhedra having only vertices of degree 3 or ℓ is at most $\log 5/\log 7$. Motivated by work of Ewald, we show that polyhedral quadrangulations with maximum degree at most 5 have shortness coefficient at most $30/37$. Finally, we define path analogues for shortness parameters, and propose first dependencies between these measures.

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1 Introduction

In this paper, in the light of Steinitz' Theorem, a *polyhedron* shall be a plane 3-connected graph. (A graph is *plane* if it is planar and embedded in the Euclidean plane.) We denote by $\mathcal{P}_{k,\ell}$ the family of all polyhedra in which every vertex has degree k or ℓ. If $k = \ell$, we simply write $\mathcal{P}_k$. Given an infinite family $\mathcal{G}$ of graphs and a function $f : \mathcal{G} \to \mathbb{R}$, $\liminf_{G\in\mathcal{G}} f(G)$ is defined to be $\liminf_{i\to\infty} f(G_i)$, where we take an (arbitrary) enumeration of graphs in $\mathcal{G} = \{G_1, G_2, \dots\}$. We shall study the *shortness coefficient* and *shortness exponent* of an infinite family $\mathcal{G}$ of graphs, defined as

$$\rho(\mathcal{G}) := \liminf_{G\in\mathcal{G}} \frac{\operatorname{circ}(G)}{|V(G)|} \qquad \text{and} \qquad \sigma(\mathcal{G}) := \liminf_{G\in\mathcal{G}} \frac{\log \operatorname{circ}(G)}{\log |V(G)|},$$

respectively, where the *circumference* $\operatorname{circ}(G)$ denotes the length of a longest cycle in a given graph $G := (V(G), E(G))$. For a classical reference, see [5].

It is well-known that, unlike planar 4-connected graphs, polyhedra can have relatively short longest cycles. Building on the influential work of Moon and Moser [14], Chen and Yu [3] showed that the shortness exponent of polyhedra is $\log_3 2$. This implies that the shortness coefficient of polyhedra is 0. Grünbaum and Walther [5] were among the first to investigate the shortness parameters of polyhedra. They sought to refine the bounds by studying polyhedra within certain subclasses. Motivated by this, we propose several new upper bounds for the shortness parameters of regular polyhedra (where regularity is understood in the graph-theoretical sense) and of polyhedra with exactly two vertex degrees. We also examine the relationships between these bounds. Grünbaum and Walther showed that *quartic* (i.e. 4-regular) polyhedra have shortness exponent at most $\log 22/\log 23$. This was subsequently improved by Harant [6] to $\log 16/\log 17$, which holds even when all faces are either triangles or of length k, for infinitely many k. In this article we complement Harant's result by strengthening the Grünbaum-Walther bound to $\log 4/\log 5$, and showing that this bound even holds for the family of quartic polyhedra with faces of length at most 7.

We also prove that for every $4 \le \ell \le 8$ the shortness exponent of the polyhedra having only vertices of degree 3 or ℓ is at most $\log 5/\log 7$. Motivated by work of Ewald [4] and Reynolds [16], we show that polyhedral *quadrangulations*—plane graphs in which all faces have length 4—with maximum degree at most 5 have shortness coefficient at most $30/37$. A good overview (and further references) regarding results on shortness parameters in quadrangulations (and triangulations) with restricted vertex degrees is given by Jendrol' and Kekeňák in [10]. Finally, we define path analogues for shortness parameters, and propose first dependencies between these measures.

We introduce further terminology used throughout this article. A *k-leg fragment* F shall be a graph that contains exactly k half-edges, which we also call the *legs* of F. Such a fragment F may be constructed from an induced subgraph $G[V(F)]$ of a graph G by adding exactly the edges as legs that have one end-vertex in $G[V(F)]$ and the other end-vertex in $G[V(G) \setminus V(F)]$. Consider a k-leg fragment F and a graph G containing a k-leg fragment W. We say that we *replace W with F* in G if W together with its k legs is removed from G, leaving k dangling half-edges, to which we connect the k legs of F using a bijection. Throughout this paper, if $G - W$ and F are plane, it will always be possible to choose a bijection such that replacing W with F yields a plane graph as well.

If G contains vertex-disjoint copies of W, we denote by $F \xrightarrow{W} G$ the graph obtained by replacing *every* copy of W in G with F. In a plane 2-edge-connected graph, the *length* of a face is the number of edges in its boundary. The join of K_1 and a cycle shall be called a *wheel*, an *n-wheel* is a wheel of order n.

2 Shortness exponents of polyhedral graphs with few distinct vertex degrees

We first consider shortness exponents and start with a generalisation of a method proposed by Bielig in [1]; see also [17, Lemma 2].

Lemma 1. *Let F and W be k-leg fragments such that F contains $z \geq 2$ vertex-disjoint copies of W. Let G_0 be a graph that contains W, and define $G_i := F \xrightarrow{W} G_{i-1}$ for every $i > 0$. If for every cycle C of any graph that contains F as an induced subgraph, either C is contained in F or C misses at least $w \leq z-2$ copies of W in F, then $\sigma(\{G_i\}_{i\geq 0}) \leq \log(z-w)/\log z$.*

Proof. Suppose G_0 contains $z' > 0$ vertex-disjoint copies of W. Then G_i contains $z^i z'$ copies of W for every $i \geq 0$ and $z^{i-1}z'$ copies of F for every $i \geq 1$. Hence, G_i has at least $z^i z'|V(W)|$ vertices. On the other hand, by induction, every cycle C_i of G_i for $i \geq 0$ has length at most

$$|V(G_0)| + \sum_{j=0}^{i-1}(z-w)^j z'|V(F)| = |V(G_0)| + \frac{(z-w)^i - 1}{(z-w)-1} \cdot z'|V(F)|,$$

whence

$$\begin{aligned}\sigma(\{G_i\}_{i\geq 0}) &= \lim_{i\to\infty} \frac{\log \operatorname{circ}(G_i)}{\log |V(G_i)|} \\ &\leq \lim_{i\to\infty} \frac{\log\left(|V(G_0)| + \frac{(z-w)^i-1}{(z-w)-1} \cdot z'|V(F)|\right)}{\log(z^i z'|V(W)|)} = \frac{\log(z-w)}{\log z}.\end{aligned}$$

□

2.1 Upper bound for $\sigma(\mathcal{P}_4)$

Consider the graph H having circumference 34 in Figure 1 and its vertices x, x_1, x_2, x_3 and x_4. Remove x from H and obtain a 4-leg fragment F. Consider any graph G containing F as an induced subgraph such that only x_1, x_2, x_3, x_4 have a neighbour outside F, and moreover each x_i has exactly one such neighbour (but these neighbours are allowed to coincide). Let C be a cycle of G that is not entirely contained in F but contains at least one vertex of F.

We now prove that C visits at most 33 of the 38 vertices of F, and that there exists a 5-wheel in F such that C has an empty intersection with this 5-wheel.

First, observe that if C contains a vertex of a 5-wheel W in F, we can modify C to be another cycle that includes $V(C) \cup V(W)$, while leaving C unchanged outside F. Therefore, we can assume without loss of generality that if C intersects a 5-wheel, it contains the entire wheel.

Assume by contradiction that $|V(C) \cap V(F)| \geq 34$, or that C intersects all the 5-wheels of F. We will omit the symmetric cases.

If $C \cap F$ consists of one path, let a and b be the end-vertices of this path. Then $a, b \in \{x_1, x_2, x_3, x_4\}$, so let $C' := (C \cap F) \cup axb$. This forms a cycle of length $|V(C) \cap V(F)| + 1$ in H.

After making slight modifications if necessary, we can assume that $V(C') \supset \{x_1, x_2, x_3, x_4\}$. If $|V(C) \cap V(F)| \geq 34$, then C' has length at least 35. On the other hand, if C intersects all the 5-wheels, then C' contains all the vertices of the six 5-wheels in H, and thus its length would be 36. In both cases, this contradicts the fact that H has circumference 34.

If $C \cap F$ consists of two paths, let P and Q be these two paths. If P is an x_1x_2-path and Q is an x_3x_4-path, then the cycle $P \cup Q \cup x_1x_4 \cup x_2x_3$ is a cycle of length at least 35 in H if $|V(C) \cap V(F)| \geq 34$, or at least 36 if C intersects all the 5-wheels. This leads to a contradiction. The same argument applies if P is an x_1x_4-path and Q is an x_2x_3-path. Since F is planar, no other situations can occur.

Therefore, $|V(C) \cap V(F)| \leq 33$. Let W be any of the five 5-wheels contained in F. Figure 1 shows that, if $C \cap F$ contains a vertex of W, $C \cap F$ can be extended to a path or a vertex-disjoint union of two paths with the same end-vertices that contains all vertices of W. Combining this with the result above that C visits at most 33 vertices of F, we conclude that there is a 5-wheel W in F such that $C \cap W = \emptyset$. A very similar reasoning for H yields that also a longest cycle of H misses one entire 5-wheel. We summarise our findings in the following lemma and use this to prove our new bound on $\sigma(\mathcal{P}_4)$.

Lemma 2. *Consider the graph H and its vertex x in Figure 1, and let G be a graph containing the fragment $F := H - x$ as an induced subgraph such that the only vertices of F with neighbours in $G - F$ are in $N(x)$. For any cycle C in G, there is an induced 5-wheel W in F such that $C \cap W = \emptyset$; in particular, this holds for $G = H$.*

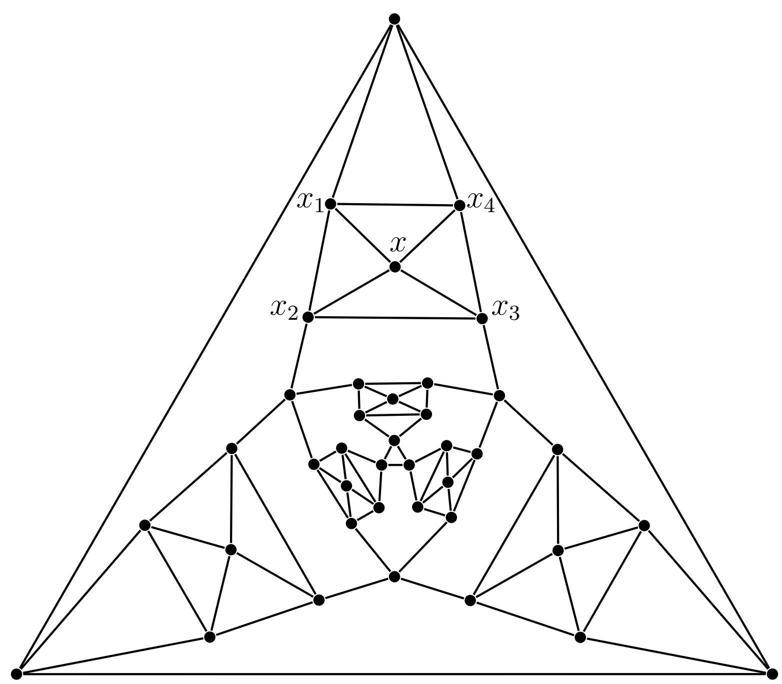

Figure 1: The quartic polyhedron H of order 39 and circumference 34, see [18]. The 4-leg fragment F is obtained from H by removing x.

Theorem 3. *The shortness exponent of the family of all quartic polyhedra, and of the family of all quartic polyhedra in which every face has length at most* 7*, is at most* $\log 4/\log 5 \approx 0.861353$.

Proof. Consider the graph H and the 4-leg fragment F from Lemma 2; F contains $z := 5$ vertex-disjoint copies of the 5-wheel W. We take $G_0 := H$, and define $G_i := F \xrightarrow{W} G_{i-1}$ for $i > 0$. As shown above, every cycle that is not entirely contained in F misses at least $w := 1$ copy of W. By Lemma 1, this implies that $\sigma(\{G_i\}_{i\geq 0}) \leq \log 4/\log 5$. Finally, one can easily check that every G_i is a quartic polyhedron and that every face of G_i has length at most 7 (this gives the second claim). As this class is a subclass of the quartic polyhedra, the first claim follows from the second. □

Theorem 6 (ii)–(iv) of Grünbaum and Walther [5] states that the shortness exponent for the family of all quartic polyhedra, with face lengths bounded above by 24, 26, and 30, respectively, is at most $\frac{\log 26}{\log 27} \approx 0.988549$, $\frac{\log 24}{\log 25} \approx 0.987318$, and $\frac{\log 22}{\log 23} \approx 0.985823$. The best-known bound on the shortness exponent for general quartic polyhedra is $\frac{\log 16}{\log 17} \approx 0.978602$, which was given by Harant [6]. Theorem 3 improves all of these bounds.

If only two types of faces are allowed, Harant [6] (see also [9]) has shown that the shortness exponent of the family of all quartic polyhedra containing only triangles and faces of length k for any $k \geq 58$ such that k is not a multiple of 3 is at most $\log 16/\log 17 \approx 0.978602$. Owens [15] has proven that the shortness exponent of the family of all quartic polyhedra containing only triangles and faces of length k is less than 1 for any $k \geq 12$. Note that, since for a given infinite family $\mathcal{G}$ of graphs, $\sigma(\mathcal{G}) < 1$ implies $\rho(\mathcal{G}) = 0$, the shortness coefficient of the family of all quartic polyhedra (and various subclasses thereof) is 0.

2.2 Upper bound for $\sigma(\mathcal{P}_{3,\ell})$ for every $4 \leq \ell \leq 8$

Let H be the 14-vertex Fruchard graph or one of the four derived 14-vertex graphs from Figure 2. The vertex set is the same for all these graphs and can be partitioned into V_1 and V_2 such that V_1 and V_2 have eight and six vertices, respectively, and $\bigcup_{v\in V_1} N(v) \subseteq V_2$. This partition is depicted in Figure 2, where the white and the black vertices constitute V_1 and V_2, respectively.

Take an arbitrary vertex $v \in V_1$, and consider the 3-leg fragment F obtained from H by deleting v. In particular, F has seven *white vertices* and six *black vertices*. Let G be a graph containing F as an induced subgraph, and let C be a cycle of G. Since only black vertices are incident to a half-edge in F and $\bigcup_{w\in V_1} N(w) \subseteq V_2$, we conclude that the cycle C misses at least two vertices of $V_1 - v$ if it is not entirely contained in F.

Theorem 4. *For every* $4 \leq \ell \leq 8$*, the shortness exponent of the family* $\mathcal{P}_{3,\ell}$ *is at most* $\log 5/\log 7 \approx 0.827087$.

Proof. Applying Lemma 1 gives the claim when choosing $G_0 := H$, F as the 3-leg fragment obtained from H by removing an arbitrary vertex $v \in V_1$, and W as a single vertex of degree 3 in $V_1 - v$. □

2.3 Relations between shortness exponents

For two graph classes $\mathcal{G}_1$ and $\mathcal{G}_2$ that satisfy $\mathcal{G}_1 \subseteq \mathcal{G}_2$, $\sigma(\mathcal{G}_1) \geq \sigma(\mathcal{G}_2)$ holds by definition. In the following, we derive more detailed relations between the shortness exponents of subclasses of polyhedra. We first prove a strengthening of a result of Bielig [1] (see also [17]).

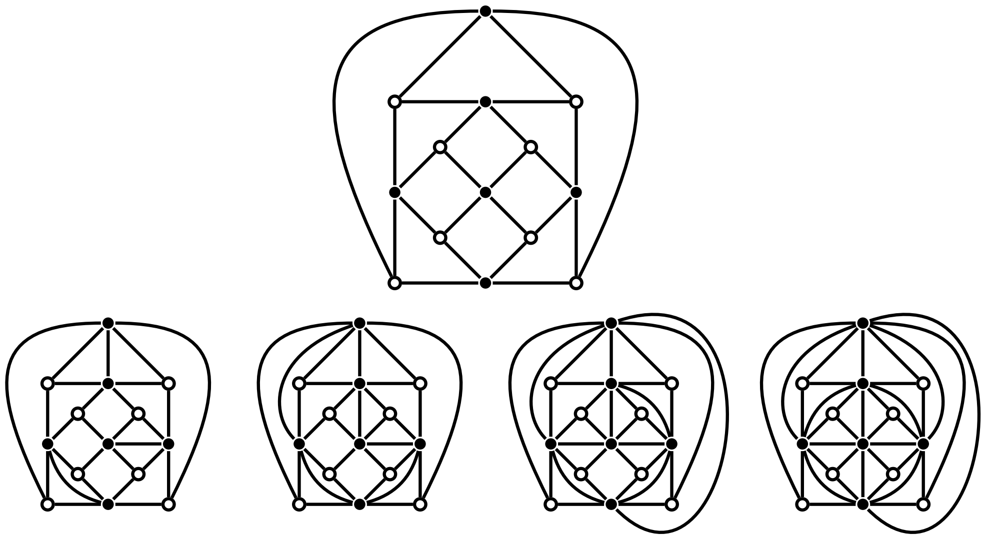

Figure 2: The Fruchard graph (top) and four derived graphs (bottom). Each derived graph is obtained by iteratively adding a matching that covers precisely all black vertices. These are all non-traceable polyhedra on 14 vertices with degrees 3 and 4 (resp. 3 and 5, 3 and 6, 3 and 7, 3 and 8). Note that the white vertices form an independent set and all have degree 3.

Lemma 5. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be infinite families of graphs and $\varphi : \mathcal{G}_1 \to \mathcal{G}_2$. If there exist constants $a, b > 0$ such that every $G \in \mathcal{G}_1$ satisfies $\operatorname{circ}(\varphi(G)) \leq a \cdot \operatorname{circ}(G)$ and $|V(\varphi(G))| \geq b \cdot |V(G)|$, then $\sigma(\mathcal{G}_1) \geq \sigma(\mathcal{G}_2)$.*

Proof. Define $\mathcal{G}_2' := \{\varphi(G) : G \in \mathcal{G}_1\}$. Clearly, $\mathcal{G}_2' \subseteq \mathcal{G}_2$. For every $H \in \mathcal{G}_2'$, there are only finitely many graphs $G \in \mathcal{G}_1$ with $\varphi(G) = H$, since $|V(G)| \leq b^{-1} \cdot |V(H)|$. This implies that $\mathcal{G}_2'$ is infinite and that

$$\sigma(\mathcal{G}_2') = \liminf_{G \in \mathcal{G}_2'} \frac{\log \operatorname{circ}(G)}{\log |V(G)|} = \liminf_{G \in \mathcal{G}_1} \frac{\log \operatorname{circ}(\varphi(G))}{\log |V(\varphi(G))|}.$$

Since a and b are constants and any sequence of graphs $\varphi(G)$ with $G \in \mathcal{G}_1$ has an unbounded number of vertices, we have

$$\liminf_{G \in \mathcal{G}_1} \frac{\log(a^{-1} \cdot \operatorname{circ}(\varphi(G)))}{\log(b^{-1} \cdot |V(\varphi(G))|)} = \liminf_{G \in \mathcal{G}_1} \frac{\log \operatorname{circ}(\varphi(G))}{\log |V(\varphi(G))|}.$$

This gives the claim that

$$\begin{aligned} \sigma(\mathcal{G}_1) = \liminf_{G \in \mathcal{G}_1} \frac{\log \operatorname{circ}(G)}{\log |V(G)|} &\geq \liminf_{G \in \mathcal{G}_1} \frac{\log(a^{-1} \cdot \operatorname{circ}(\varphi(G)))}{\log(b^{-1} \cdot |V(\varphi(G))|)} \\ = \liminf_{G \in \mathcal{G}_1} \frac{\log \operatorname{circ}(\varphi(G))}{\log |V(\varphi(G))|} &= \sigma(\mathcal{G}_2') \geq \sigma(\mathcal{G}_2). \end{aligned}$$

□

Theorem 6. *The following inequalities hold.*

1. $\sigma(\mathcal{P}_3) \geq \sigma(\mathcal{P}_4)$;
2. $\sigma(\mathcal{P}_{3,\ell}) \geq \sigma(\mathcal{P}_{5,\ell})$ *for every* $\ell \geq 3$;
3. $\sigma(\mathcal{P}_3) \geq \sigma(\mathcal{P}_5)$; *and*
4. $\sigma(\mathcal{P}_3) \geq \sigma(\mathcal{P}_{4,k})$ *for every* $k \geq 3$.

Proof.

1. For every $G \in \mathcal{P}_3$, let $\varphi(G)$ be the medial graph (or, here equivalently, the line graph) of G. It is clear that $\varphi(G) \in \mathcal{P}_4$, and $|V(\varphi(G))| = \frac{3}{2}|V(G)|$, since, by definition, each element in $V(\varphi(G))$ corresponds to an element in $E(G)$. Moreover, every vertex v of G corresponds to the boundary cycle T_v of a 3-face in $\varphi(G)$ such that the set $V(T_v)$ corresponds to $\{vw \in E(G) : w \in N_G(v)\}$, and this partitions the edges of $\varphi(G)$ into sets of size 3.

   Let C be a longest cycle of $\varphi(G)$. We want to translate this cycle into a (long) cycle in G. There is a canonical way to do this translation, but this translation only works if there is no vertex $v \in V(G)$ such that $|C \cap T_v| = 2$. Therefore, we will first translate C into another cycle which satisfies this condition. Note that since G contains a cycle of length at least 3, we have that C has at least six vertices. For any vertex $v \in V(G)$, the intersection $C \cap T_v$ contains thus at most two edges. If it contains exactly two edges, we can replace these two edges by the third edge from T_v to obtain a shorter cycle. Exhaustively iterating this process, we obtain a cycle C' in $\varphi(G)$ with $|V(C)| \leq 2 \cdot |V(C')|$. It is easily seen that G contains a cycle of length at least $|V(C')|$, hence we have that $\mathrm{circ}(\varphi(G)) \leq 2 \cdot \mathrm{circ}(G)$ and, by Lemma 5, $\sigma(\mathcal{P}_3) \geq \sigma(\mathcal{P}_4)$.

2. For every $G \in \mathcal{P}_{3,\ell}$, we replace every cubic vertex $v \in V(G)$ by the 3-leg fragment shown in Figure 3 to obtain a graph $\varphi(G) \in \mathcal{P}_{5,\ell}$. Clearly, $|V(\varphi(G))| \geq |V(G)|$. By transforming cycles in $\varphi(G)$ of length at least 16 into cycles in G, one can show that $\mathrm{circ}(\varphi(G)) \leq 15 \cdot \mathrm{circ}(G)$, and hence, by Lemma 5, $\sigma(\mathcal{P}_{3,\ell}) \geq \sigma(\mathcal{P}_{5,\ell})$.

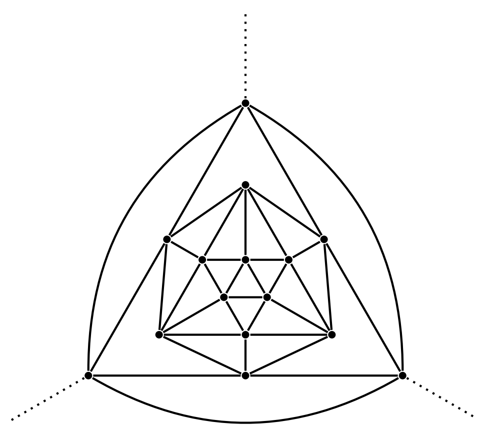

Figure 3: A 15-vertex 3-leg fragment.

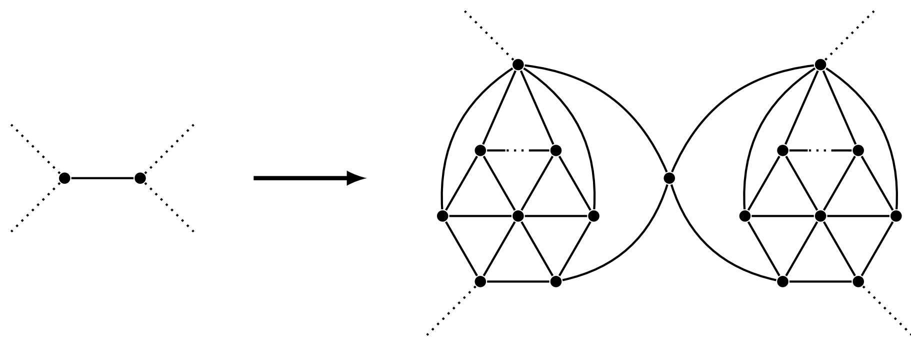

Figure 4: Replacing each edge in a perfect matching with the fragment shown on the right. Each replacement results in four vertices of degree k and $2k+1$ vertices of degree 4.

3. This case is analogous to the previous case, but here we replace all vertices by the 3-leg fragment of Figure 3.
4. Let $G \in \mathcal{P}_3$. Consider a perfect matching M in G, which must exist by Petersen's Theorem. If we replace each edge of M with the fragment shown in Figure 4, we obtain a graph, denoted $\varphi(G)$, which belongs to $\mathcal{P}_{4,k}$. Clearly, $|V(\varphi(G))| \geq |V(G)|$. Let C be a longest cycle in $\varphi(G)$. It is clear that C is not contained in any fragment. We can transform C into a cycle C' in G as follows: the edges not contained in any fragment are retained. The other edges form a vertex-disjoint family of paths, and each path is contracted to a vertex or an edge in the obvious way. Since the length of C is at most $k+3$ times the length of C', we have $\text{circ}(\varphi(G)) \leq (k+3) \cdot \text{circ}(G)$. Hence, by Lemma 5, we conclude that $\sigma(\mathcal{P}_3) \geq \sigma(\mathcal{P}_{4,k})$.

This gives the claim. □

We conclude this section with an overview of the best currently available bounds (see Table 1) that arise from Theorems 3, 4, and 6.

3 Shortness coefficient for polyhedral quadrangulations with maximum degree at most 5

Ewald [4] proved that the polyhedra with both maximum degree and face lengths at most 4 have shortness exponent 1, and it is easy to see that they need not be hamiltonian by considering for instance Fruchard's graph depicted in Fig. 2, or Herschel's graph—see also Reynolds' paper [16]. However, no upper bound less than 1 is known for the shortness coefficient. Continuing our investigation of the shortness parameters of polyhedral graphs with few distinct vertex degrees, we now prove an upper bound for the polyhedra with maximum degree at most 5 in which all faces have length 4.

Consider the 37-vertex fragment F of Figure 5 (a). By connecting the legs on the top of F one-by-one with the ones on the bottom from left to right, we obtain a cylinder C that may be extended arbitrarily by joining horizontally i further copies of C. Let G_i be

	Best currently available bounds for the shortness exponent:		
Graph class	Lower bounds	Matching bound	Upper bounds
$\mathcal{P}_3$	$\max(\sigma(\mathcal{P}_4), \sigma(\mathcal{P}_5))$		$\frac{\log 22}{\log 23}$
$\mathcal{P}_4$	$\frac{\log 2}{\log 3}$		$\min(\sigma(\mathcal{P}_3), \frac{\log 4}{\log 5})$
$\mathcal{P}_5$		$\sigma(\mathcal{P}_{3,5})$	
$\mathcal{P}_{3,4}$	$\sigma(\mathcal{P}_{4,5})$		$\min(\sigma(\mathcal{P}_4), \frac{\log 5}{\log 7})$
$\mathcal{P}_{3,5}$	$\sigma(\mathcal{P}_{4,5})$		$\min(\sigma(\mathcal{P}_3), \frac{\log 5}{\log 7})$
$\mathcal{P}_{3,6}$	$\frac{\log 2}{\log 3}$		$\min(\sigma(\mathcal{P}_3), \frac{\log 5}{\log 7})$
$\mathcal{P}_{3,7}$	$\frac{\log 2}{\log 3}$		$\min(\sigma(\mathcal{P}_3), \frac{\log 5}{\log 7})$
$\mathcal{P}_{3,8}$	$\frac{\log 2}{\log 3}$		$\min(\sigma(\mathcal{P}_3), \frac{\log 5}{\log 7})$
$\mathcal{P}_{4,5}$	$\frac{\log 2}{\log 3}$		$\min(\sigma(\mathcal{P}_5), \sigma(\mathcal{P}_{3,4}))$

Table 1: An overview of the bounds for shortness exponents for several subclasses of polyhedra. The lower bounds $\frac{\log 2}{\log 3}$ in this table arise from the general lower bound for polyhedra [3]. The upper bound $\frac{\log 22}{\log 23}$ in the top row is implied by the first statement of Section 2.3 and [5, Theorem 7(iv)]. The matching bound $\sigma(\mathcal{P}_{3,5})$ is implied by combining the same argument with $\sigma(\mathcal{P}_{3,5}) \geq \sigma(\mathcal{P}_5)$, which follows from Theorem 6.2 for $\ell = 5$. This matching bound in turn implies the lower bound $\sigma(\mathcal{P}_{3,5}) = \sigma(\mathcal{P}_5) \geq \sigma(\mathcal{P}_{4,5})$. Using Theorem 6.2 for $\ell = 4$ gives the lower bound $\sigma(\mathcal{P}_{3,4}) \geq \sigma(\mathcal{P}_{4,5})$.

the graph obtained from this extended cylinder by joining the remaining half-edges at each end to a copy of the graph shown in Figure 5 (b), which we will call a *cap*, via a bijection given by the order of the legs as given in Figure 5 (so that planarity is preserved). For the half-edges at the right-hand end of C the bipartition shown in Figure 5 (b) can be used, while for the half-edges at the left-hand end of C we reverse black and white in order to obtain a vertex-colouring, which proves that G_i is bipartite.

Clearly, the vertices of the caps do not influence the shortness coefficient. Since F is bipartite and has exactly 15 black vertices and 22 white vertices, we obtain the following theorem.

Theorem 7. *The shortness coefficient of the polyhedral quadrangulations with maximum degree at most* 5 *is at most* $30/37 = 0.\overline{810}$*.*

Note that this complements the corresponding shortness exponent result in the list given by Grünbaum and Walther [5].

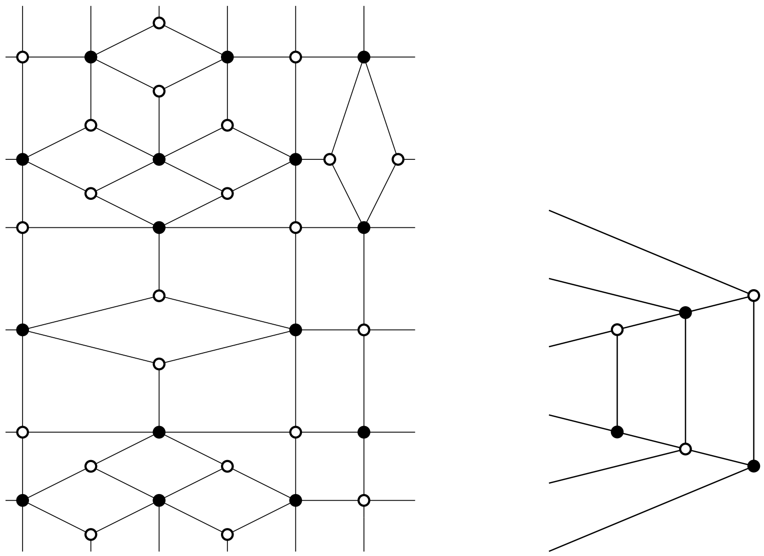

Figure 5: (a) A 37-vertex fragment. (b) A cap.

4 Shortness parameters for paths

In this brief closing section, we extend the results known for shortness coefficients and exponents by proposing similar measures for longest paths, and relate these measures to the well-known ones. Following Kapoor, Kronk, and Lick [11], for a graph G, let $\partial(G)$ denote the length (i.e. number of edges) of a longest path in G. We now define the *path shortness coefficient* and the *path shortness exponent* of a family $\mathcal{G}$ of graphs as

$$\rho^P(\mathcal{G}) := \liminf_{G\in\mathcal{G}} \frac{\partial(G)}{|V(G)|} \qquad \text{and} \qquad \sigma^P(\mathcal{G}) := \liminf_{G\in\mathcal{G}} \frac{\log \partial(G)}{\log |V(G)|},$$

respectively.

Clearly, long cycles imply long paths. All known lower bounds for the shortness coefficient and exponents are also valid for the path coefficient and exponent, as replacing $\mathrm{circ}(G)$ with $\mathrm{circ}(G) - 1$ in the numerator does not influence the limit. The validity of the converse direction is in general not known. However, if we have information on the connectedness of the graphs, we have the following.

Theorem 8. *Let $k \geq 3$ and $\mathcal{G}$ be an infinite family of k-connected graphs. Then*

1. $\rho^P(\mathcal{G}) \leq \frac{3k-4}{2k-4} \cdot \rho(\mathcal{G})$;
2. $\sigma^P(\mathcal{G}) = \sigma(\mathcal{G})$; *and*
3. *if $k = 3$ and every graph of $\mathcal{G}$ is 3-regular,* $\rho^P(\mathcal{G}) \leq \frac{3}{2} \cdot \rho(\mathcal{G})$.

Proof. Locke [13] proved that if a k-connected graph G contains a path of length ℓ, then G contains a cycle of length at least $\frac{2k-4}{3k-4} \cdot \ell$. This implies the first two statements of the theorem. For $k = 3$ and 3-regular graphs G, Bondy and Locke [2] showed the stronger result that every path of length ℓ in G implies that G contains a cycle of length at least $\frac{2}{3} \cdot \ell + 2$; since constant summands do not influence the limit, the third statement follows. □

Applying Theorem 8 to well-known results for shortness coefficients gives the following corollary.

Corollary 9. *Every subclass of* 3*-connected graphs with shortness exponent less than* 1 *has path shortness coefficient* 0*, for example:*

1. *the cubic, quartic and quintic polyhedra by [5, Theorem A], Theorem 3 and [5, Theorem 5(i)];*
2. *for every* $q \geq 136$*, the* 3*-regular polyhedral graphs in which every face has length at most* q *by [8, Theorem 2];*
3. *for every* $q \geq 29$ *such that* $q \not\equiv 0 \pmod 3$*, the* 5*-regular polyhedral graphs in which every face has either length* 3 *or* q *by [7, p. 144]; and*
4. *the* 5*-connected* 1*-planar graphs by [12, Section 3.2].*

References

[1] G. Bielig. Untersuchungen über maximale Kreise auf 3-polytopalen Graphen. PhD Thesis, Bochum, FRG, 1976.

[2] J. A. Bondy and S. C. Locke. Relative Lengths of Paths and Cycles in 3-Connected Graphs. *Discrete Math.* **33** (1981) 111–122.

[3] G. Chen and X. Yu. Long Cycles in 3-Connected Graphs. *J. Combin. Theory Ser. B* **86** (2002) 80–99.

[4] G. Ewald. On Shortness Exponents of Families of Graphs. *Israel J. Math.* **16** (1973) 53–61.

[5] B. Grünbaum and H. Walther. Shortness exponents of families of graphs. *J. Combin. Theory Ser. A* **14** (1973) 364–385.

[6] J. Harant. Über den Shortness Exponent regulärer Polyedergraphen mit genau zwei Typen von Elementarflächen. PhD Thesis, Ilmenau, GDR, 1982.

[7] J. Harant, P. J. Owens, M. Tkáč, and H. Walther. 5-regular 3-polytopal graphs with edges of only two types and shortness exponents less than one. *Discrete Math.* **150** (1996) 143–153.

[8] J. Harant and H. Walther. On a Problem Concerning Longest Circuits in Polyhedral Graphs. *Ann. Discrete Math.* **41** (1989) 211–219.

[9] J. Harant and H. Walther. Some new results about the shortness exponent in polyhedral graphs. *Časopis pro pěstování matematiky* **112** (1987) 114–122.

[10] S. Jendroľ and R. Kekeňák. Longest circuits in triangular and quadrangular 3-polytopes with two types of edges. *Math. Slovaca* **40** (1990) 341–357.

[11] S. F. Kapoor, H. V. Kronk, and D. R. Lick. On detours in graphs. *Canad. Math. Bull.* **11** (1968) 195–201.

[12] O.-H. S. Lo, J. M. Schmidt, N. Van Cleemput, and C. T. Zamfirescu. Shortness Coefficient of Cyclically 4-Edge-Connected Cubic Graphs. *Electron. J. Combin.* **27** Iss. 1 (2020), Article Nr. P1.43.

[13] S. C. Locke. Relative Lengths of Paths and Cycles in k-Connected Graphs. *J. Combin. Theory Ser. B* **32** (1982) 206–222.

[14] J. W. Moon and L. Moser. Simple paths on polyhedra. *Pacific J. Math.* **13** (1963) 629–631.

[15] P. J. Owens. Regular planar graphs with faces of only two types and shortness parameters. *J. Graph Theory* **8** (1984) 253-275.

[16] C. N. Reynolds. Circuits upon polyhedra. *Ann. Math.* **33** (1932), 367-372.

[17] M. Schmidt and T. Zamfirescu. Shortness exponents for polytopes which are k-gonal modulo n. *J. Combin. Theory Ser. B* **33** (1982) 101–120.

[18] N. Van Cleemput and C. T. Zamfirescu. Regular non-hamiltonian polyhedral graphs. *Appl. Math. Comput.* **338** (2018) 192–206.

[19] H. Walther. Longest cycles in polyhedral graphs. *Israel J. Math.* **83** (1993) 203–212.