Cycle spectra of contraction-critically 4-connected planar graphs

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Abstract

Motivated by the long-standing and wide open pancyclicity conjectures of Bondy and Malkevitch, we study the cycle spectra of contraction-critically 4-connected planar graphs. We show that every contraction-critically 4-connected planar graph on n vertices contains a cycle of length k for every $k \in \{\lfloor \frac{n}{2} \rfloor - \lceil \frac{n}{108} \rceil, \dots, \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{36} \rfloor\} \cup \{\frac{2}{3}n, \dots, n\}$.

1 Introduction

A fundamental question in graph theory asks for the set of lengths of cycles in a graph; this set is called the *cycle spectrum* of the graph. For instance, hamiltonicity is one of the special cases in this spectrum that received tremendous attention. A graph G is *pancyclic* if G contains a k-cycle for every $3 \le k \le n$, where a *k-cycle* is a cycle of length k.

Tutte [17] showed in 1956 that every 4-connected planar graph is hamiltonian. In 1973, Bondy [2] noticed that many non-trivial sufficient conditions for hamiltonicity imply pancyclicity. He made the following conjecture (see also [1, Conjecture 2]).

Conjecture A (Bondy [2], 1973)**.** Every 4-connected planar graph contains cycles of all lengths $3 \le k \le n$, with the possible exception of one even length.

The reason for the exception of one even length is due to the fact that there are infinitely many 4-connected planar graphs that have no cycle of length four [9] (e.g. the line graphs of 3-regular cyclically 4-edge-connected planar graphs with girth five). A similar conjecture was proposed by Malkevitch (we note that Malkevitch made a weaker conjecture restricted to 4-regular graphs in 1976 [10, Conjecture 1]).

Conjecture B (Malkevitch [11, Conjecture 6.1], 1988)**.** Every 4-connected planar graph that contains a 4-cycle is pancyclic.

Today, both conjectures of Bondy and Malkevitch are still wide open. Indeed, the best results so far assure only constantly many different cycle lengths. By using the discharging method, it was proved that every planar graph with minimum degree at least four contains cycles of length 3 (Euler's formula), 5 [18] and 6 [5]. For lengths from the other end of the cycle spectrum, the machinery of Tutte cycles [17] has been used. A series of work [13, 16, 15, 14, 3, 4] showed that every 4-connected planar graph on n vertices contains a k-cycle for every $k \in \{n-7, \dots, n-1\}$ with $k \ge 3$.

Recently, the first author [8] proved that every planar hamiltonian graph with minimum degree at least four contains a k-cycle for every $k \in \{\lfloor \frac{n}{2} \rfloor, \dots, \lceil \frac{n}{2} \rceil + 3\}$. While this is the first result about

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cycle lengths that are not at an end of the spectrum, it provides only a constant number of cycle lengths here. Indeed, Chen, Fan and Yu [3] proposed the following conjecture about a constant number of cycle lengths in these graphs.

Conjecture C (Chen, Fan and Yu [3, Conjecture 1.2], 2004)**.** Every 4-connected planar graph has a k-cycle for every $k \in \{n-25, \dots, n\}$ with $k \geq 3$.

An interesting discussion in [3] reveals that Tutte's approach, which was used in proving the cycle lengths $n-3, \dots, n$, cannot be used alone for further improvements. Indeed, the cycle lengths $n-7, \dots, n-4$ were proved by a combination of Tutte's method and the distribution of 4-contractible edges in these graphs [3, 4]. This motivates us to approach the conjectures above by studying *contraction-critically* 4-connected planar graphs, i.e. the 4-connected planar graphs for which the contraction of any edge results in a graph that is not 4-connected. As contraction-critical graphs often serve as base case for inductive proofs in structural graph theory (for more details, we refer to a beautiful survey of Kriesell [7]), this may be a good starting point for attacking the aforementioned conjectures.

Our main result is to assure two linearly-sized intervals in the spectra of contraction-critically 4-connected planar graphs. In particular, we show that Conjecture C holds for (sufficiently large) contraction-critically 4-connected planar graphs in a strong sense, namely even if the number 25 is replaced by the linear term $n/3$.

Theorem 1. *Every contraction-critically 4-connected planar graph on n vertices contains a k-cycle for every $k \in \{\lfloor \frac{n}{2} \rfloor - \lceil \frac{n}{108} \rceil, \dots, \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{36} \rfloor\} \cup \{\frac{2}{3}n, \dots, n\}$.*

We will prove our main theorem in Sections 2 (as Corollary 5) and 3 (as Theorem 7). In addition, we will show in Section 2 that if G has any k-cycle with $k > 3$, then it has more than $k/3$ cycles of consecutive lengths including the length k.

We remark that our method also shows that there are *many* cycles of the lengths given above: e.g. the number of k-cycles for every $k \in \{2n/3, \dots, n\}$ is at least $\binom{n/3}{k-2n/3}$. For the sake of concise proofs, we will however restrict ourselves to prove only one cycle for each length.

Preliminaries

One of the main tools behind our proofs is the characterization of contraction-critically 4-connected graphs by Fontet [6] and Martinov [12]. They showed that a graph is contraction-critically 4-connected if and only if it is the square of a cycle of length at least five[1] or the line graph of a 3-regular cyclically 4-edge-connected[2] graph. Since the line graph of a non-planar graph is non-planar, a planar graph is contraction-critically 4-connected if and only if it is

- the square of a cycle of even length at least six or
- the line graph of a 3-regular cyclically 4-edge-connected planar graph.

In both cases, the graph is 4-regular.

A *k-face* is a face whose boundary has k edges; we also call a 3-face T a *triangular face*, a 3-cycle T a *triangle*, and will denote T often by its (boundary) vertex set.

From now on, we assume that G is a simple contraction-critically 4-connected (and hence 4-regular) planar graph on n vertices embedded in the plane. We further assume that G is not the

[1] A square of a cycle is obtained from the cycle by joining each pair of vertices at distance two with an edge.

[2] A graph G is *cyclically 4-edge-connected* if, for every edge-cut S of G with less than 4 edges, $G - S$ has a component that contains no cycle.

square of a cycle, as these graphs are easily shown to be pancyclic. Hence, G is the line graph of a (unique) 3-regular cyclically 4-edge-connected planar graph G' and G is not the octahedral graph (as that is the square of a cycle of length six), so that $G' \neq K_4$.

We claim that G' has no triangular face. Suppose G' has a triangular face with facial cycle T, then T is connected to $G' - T$ with precisely three edges, as G' is 3-regular. If these three edges have three distinct neighbors in $G' - T$, they form a 3-edge-cut that splits G' into two components each of which has some cycle, which contradicts that G' is cyclically 4-edge-connected. Likewise, it is not possible that these three edges have two neighbors in $G' - T$. Finally, if these three edges have a common neighbor in $G' - T$, then $G' = K_4$, which is however excluded from our consideration. This proves the claim.

As G is 4-regular, we have $|E(G)| = 2n$, so that G has $n + 2$ faces by Euler's formula. Given an embedding of G' in the plane, we may draw an embedding of G in the same plane such that each vertex of G is placed in its corresponding edge of G' and the embedding of G' does not intersect the embedding of $E(G)$. Hence there are $|V(G')|$ facial triangles in G, and each of them contains a single vertex of G' in the interior. By the claim of the previous paragraph, the remaining facial cycles of G, i.e. the outer cycle and those facial cycles whose interiors do not contain any vertex of G' have length greater than three. Indeed, each of these two classes of facial cycles partitions the edge set $E(G)$. We conclude that G is 2-face-colorable (equivalently, the dual of G is bipartite) and has precisely $\frac{2}{3}n$ edge-disjoint facial triangles.

Another tool we will use comes from the following result given by the first author [8]. Let C be a cycle of G. We denote by C_{int} and C_{ext} the subgraphs obtained from G by deleting the edges of G in the strict exterior and interior of C of the planar embedding of G.

Lemma 2 ([8, Lemma 1])**.** *Let $w, g, N_1, N_2 \in \mathbb{N}$ and $h \in \mathbb{Z}$ such that $w + h > 2$, $2N_1 \geq N_2 + h$, $1 \leq w \leq N_2$ and $2w - 4g - h + 3 \leq N_2 \leq 2w + g + h - 2$. Let D be a tree of N_1 vertices and let $c : V(D) \to \mathbb{N}$ be vertex weights. Define $c(S) := \sum_{v \in V(S)} c(v)$ for any subtree S of D. If $c(D) = N_2$ and $c(v) \leq w$ for all $v \in V(D)$, then there exists a subtree S of D of weight $w - g + 1 \leq c(S) \leq w$.*

Corollary 3. *Let G be a plane graph on $n \geq 6$ vertices, and C be a hamiltonian cycle of G such that C_{int} contains at least $n/2$ chords of C. Then, for every $k \in \{\lfloor \frac{n}{2} \rfloor, \ldots, \lceil \frac{n}{2} \rceil + 3\}$, C_{int} contains a k-cycle, and for every $k \in \{\lceil \frac{n}{2} \rceil + 3, \ldots, \frac{3n+7}{4}\}$, there is a $k' \in \{k, \ldots, 2k - \lceil \frac{n}{2} \rceil - 3\}$ such that C_{int} contains a k'-cycle.*

Proof. Let D be the *weak dual* of C_{int}, i.e. the graph that is obtained from the dual of C_{int} by deleting the vertex corresponding to the unbounded face. It is well-known that D is a tree. For every vertex v of D, we assign a (positive integer) *weight* to v equal to the face length of v in G minus two. Let the *weight* of a subtree be the sum of the weights of its vertices. For any subtree S of D, we define $\phi(S)$ to be the symmetric difference of the facial cycles that correspond to vertices of the subtree S (here we see cycles as edge sets). This gives us a bijective mapping between the subtrees of D and the cycles in C_{int}.

We claim that a subtree S of weight w is mapped to the cycle $\phi(S)$ of length $w + 2$. We prove this by induction on $|V(S)|$. By definition, the claim holds if $|V(S)| = 1$, so suppose $|V(S)| > 1$. Let S' be obtained from S by deleting a leaf of weight $w - w'$. Then S' has weight w' and, by induction hypothesis, $\phi(S')$ has length $w' + 2$. Note that $\phi(S')$ has exactly one common edge with the facial cycle corresponding to the deleted leaf in S, which has length $w - w' + 2$ by definition. Thus, the length of $\phi(S)$ is $(w' + 2) + (w - w' + 2) - 2 = w + 2$. This proves the claim.

Observe that D has $N_1 \geq \lceil \frac{n}{2} \rceil + 1$ vertices, as C has at least $\lceil \frac{n}{2} \rceil$ chords in C_{int}. Since $\phi(D)$ is the hamiltonian cycle C, D has weight $N_2 = c(D) = n - 2$. Moreover, $c(v) \leq \lfloor \frac{n}{2} \rfloor - 2$ for any $v \in V(D)$, as otherwise $N_2 = c(D) \geq (\lfloor \frac{n}{2} \rfloor - 1) + (N_1 - 1) \geq n - 1$, which is a contradiction.

To prove the first statement, it suffices to show that for every $w \in \{\lfloor \frac{n}{2} \rfloor - 2, \dots, \lceil \frac{n}{2} \rceil + 1\}$, D has a subtree of weight w. Take $g := 1$ and $h := 4$. It is straightforward to verify that the inequalities on w, g, N_1, N_2, h in the statement of Lemma 2 hold. Therefore, by Lemma 2 and the facts given in the previous paragraph, D has a subtree of weight w.

For the second statement, we take $w := 2k - \lceil \frac{n}{2} \rceil - 5$, $g := k - \lceil \frac{n}{2} \rceil - 2$ and $h := 4$ (where $k \in \{\lceil \frac{n}{2} \rceil + 3, \dots, \frac{3n+7}{4}\}$). Again, it is straightforward to verify that the inequalities on w, g, N_1, N_2, h as well as other conditions given in the statement of Lemma 2 hold. By Lemma 2, D has a subtree of weight w' with $w' \in \{w - g + 1, \dots, w\} = \{k - 2, \dots, 2k - \lceil \frac{n}{2} \rceil - 5\}$. This implies that C_{int} has a k'-cycle for some $k' \in \{k, \dots, 2k - \lceil \frac{n}{2} \rceil - 3\}$. $\square$

In the remainder of this section, we define the notation that we need. Let C be a cycle of G. A triangle of G is called an *i-triangle* of C if it contains exactly i edges of C. Let $T = \{v, w, z\}$ be a 1-triangle with $vw \in E(C)$ and $z \notin V(C)$. We say that vw is *extendable*, as we may extend C to a longer cycle by replacing its edge vw with the path vzw of length two. Conversely, for any 2-triangle $T = \{v, w, z\}$ with $vz, zw \in E(C)$, we may use T to *short-cut* C, i.e. to obtain a shorter cycle by deleting z from C and adding the edge vw.

2 Relative cycle lengths

In this section, we discuss how cycles of different lengths may be obtained from an l-cycle of G by extending extendable edges and short-cutting 2-triangles. As an application, we will prove that G contains a k-cycle for every $2n/3 \le k \le n$.

For a cycle C, let C_1 and C_2 be the sets of edges of C that are contained in 1- and 2-triangles, respectively. Consider an edge $e = vw \in C_1$. Let $T = \{v, w, z\}$ be the (unique) triangle containing e, and T' be the other (unique) triangle containing z. We say that e is in the set C_1^j if T' is a j-triangle. We have $C_1 = C_1^0 \mathbin{\dot\cup} C_1^1 \mathbin{\dot\cup} C_1^2$.

Lemma 4. *Let C be a cycle of length $l > 3$ of G. Then $l = |C_1| + |C_2|$ and G contains a k-cycle for every $k \in I := \{l - |C_2|/2, \dots, l + |C_1^0| + |C_1^1|/2\}$. In particular, $|I| \ge l/3 + 2|C_1^0|/3 + |C_1^1|/6 + 1$.*

Proof. Since C is not a 3-cycle, every edge of C is either in a 1- or 2-triangle. We have $l = |C_1| + |C_2| = |C_1^0| + |C_1^1| + |C_1^2| + |C_2|$ and there are exactly $|C_2|/2$ 2-triangles of C. For every $k \in \{l - |C_2|/2, \dots, l\}$, short-cutting C using $l - k$ of these 2-triangles one-by-one gives a k-cycle in G; note that this procedure does not reach a 3-cycle, as otherwise an edge of G would be contained in two triangles.

Now consider any edge $vw \in C_1$. Let $T = \{v, w, z\}$ and $T' = \{z, x, y\}$ be the two triangles that contain z. We first consider the case that $vw \in C_1^2$. As T' is a 2-triangle, $z \in V(C)$ and $xz, yz \in C_2$ and hence $|C_1^2| \le |C_2|/2$.

If $vw \in C_1^0$, vw is extendable and T' is a 0-triangle. Thus, extending C with vw preserves the edge-classifications of all edges that are not in T. If vw is in C_1^1, then vw is extendable and T' is a 1-triangle. Then extending C with vw preserves the edge-classifications of all edges that are neither in T nor in T'. As $vw \in C_1^1$ if and only if $xy \in C_1^1$, we may pick one edge from each such pair $\{vw, xy\}$ of edges to form a set $(C_1^1)'$ of $|C_1^1|/2$ edges. Now, for every $k \in \{l, \dots, l + |C_1^0| + |C_1^1|/2\}$, extending C one-by-one with $k - l$ edges from $C_1^0 \cup (C_1^1)'$ yields a k-cycle in G.

Hence, G contains a k-cycle for every $k \in I$ with

$$\begin{aligned} |I| &= |C_1^0| + |C_1^1|/2 + |C_2|/2 + 1 \\ &= l/3 + 2|C_1^0|/3 + |C_1^1|/6 - |C_1^2|/3 + |C_2|/6 + 1 \end{aligned}$$

$$\geq l/3 + 2|C_1^0|/3 + |C_1^1|/6 + 1,$$

since we proved $|C_2| \geq 2|C_1^2|$. $\square$

Lemma 4 implies immediately the following result, as every 4-connected planar graph is hamiltonian [17] and has a cycle of length $\lceil n/2 \rceil$ [8].

Corollary 5. *Every contraction-critically* 4*-connected planar graph on* n *vertices contains a cycle of length* k *for every* $k \in \{2n/3, \dots, n\} \cup I$*, where* I *is a set of consecutive integers with* $|I| \geq n/6+1$ *and* $\lceil n/2 \rceil \in I$*.*

3 Medium cycle lengths

In this section, we exhibit another linearly-sized interval of the cycle spectrum of G by a similar technique.

Lemma 6. *Let* C *be a hamiltonian cycle of* G *such that* C_{int} *contains at least* $n/2$ *chords of* C *and a cycle* H *of length* $\lfloor \frac{n}{2} \rfloor + dn > 3$ *for some* $d \geq 0$*. Then the number of* 2*-triangles of* H *is larger than* $(\frac{d}{3} + \frac{1}{108})n$*.*

Proof. Since C is a hamiltonian cycle, $V(C_{\text{int}}) = V(C)$ and $V(H_{\text{int}}) = V(H)$. Since H has length at least four and every triangle encloses a triangular face, every triangle of G is either contained in H_{int} or in H_{ext}. For $0 \leq i \leq 2$, let H^i_{int} be the set of i-triangles of H that are contained in H_{int}, and let $H^{>3}_{\text{int}}$ be the set of interior faces of length larger than three in H_{int}. For every $T \in H^0_{\text{int}}$, we have $V(T) \subseteq V(H)$, so that every vertex v of T is contained in a 2-triangle of H (which in turn is contained in H_{ext}). Likewise, for any 1-triangle $\{u, v, w\}$ in H^1_{int} such that $uw \in E(H)$, v is contained in a 2-triangle of H (which is in H_{ext}). Hence, the number of 2-triangles of H is at least $\Delta_{\text{int}} + 2|H^0_{\text{int}}|$, where $\Delta_{\text{int}} := \sum_{i=0}^{2} |H^i_{\text{int}}|$ is the number of triangles in H_{int}. Therefore, it suffices to show that $\Delta_{\text{int}} + 2|H^0_{\text{int}}| > (\frac{d}{3} + \frac{1}{108})n$.

Next we show that H_{int} does not only have one interior face. Let D be the weak dual of C_{int}. As in the proof of Corollary 3, we consider the bijective mapping ϕ between the subtrees of D and the cycles in C_{int}. Recall that a subtree S of weight k is mapped to the cycle $\phi(S)$ of length $k+2$; in particular, D has weight $n-2$. Since C_{int} has at least $\lceil \frac{n}{2} \rceil + 1$ interior faces, D has at least $\lceil \frac{n}{2} \rceil + 1$ vertices. If H_{int} has only one interior face, then D has a vertex of weight $\lfloor \frac{n}{2} \rfloor + dn - 2$. Therefore, the sum of the weights of the remaining at least $\lceil \frac{n}{2} \rceil$ vertices in D is $\lceil \frac{n}{2} \rceil - dn$, which implies $d = 0$ and that C_{int} contains exactly $\lceil \frac{n}{2} \rceil$ chords of C. Moreover, as G has no two triangles sharing an edge, D has no adjacent vertices of weight one.

Hence, D is a star graph with $\lceil \frac{n}{2} \rceil$ leaves of weight one each, such that the center vertex has weight $\lfloor \frac{n}{2} \rfloor - 2$. By the definition of vertex weights of D, the center vertex has degree at most $\lfloor \frac{n}{2} \rfloor$. This implies that n must be even. More precisely, C_{int} consists of the even cycle C and $\frac{n}{2}$ 2-triangles of C, and C_{ext} consists of C and $\frac{n}{6}$ 0-triangles of C. Then there exists a facial cycle of C_{ext} that contains exactly one chord of C and is adjacent to some 0-triangle of C. However, this facial cycle has length three, since for any pair of adjacent vertices in C, one of them has degree two and the other one has degree four in C_{ext}. This is not possible, as G has no adjacent triangular faces.

Hence, H_{int} has $f > 1$ interior faces; H_{int} has in its interior triangular faces and non-triangular faces. Let D_H be the weak dual of H_{int}. Then D_H is a tree. In D_H, the vertex that corresponds to the enclosed face of an i-triangle of H has exactly $3-i$ neighbors, each of which encloses a non-triangular face. Hence, by rooting D_H at some vertex corresponding to a face from $H^{>3}_{\text{int}}$, it is readily to see that $|H_{\text{int}}^{>3}| = 2|H_{\text{int}}^{0}| + |H_{\text{int}}^{1}| + 1$. Similarly, as the vertices that correspond to triangles in H_{int}^{1} have degree two in D_H, we have $f - |H_{\text{int}}^{1}| \geq |H_{\text{int}}^{1}| + 1$. These imply that $\Delta_{\text{int}} + 2|H_{\text{int}}^{0}| = f - |H_{\text{int}}^{>3}| + 2|H_{\text{int}}^{0}| = f - |H_{\text{int}}^{1}| - 1 \geq (f-1)/2$. Thus, it suffices to show that $f > cn + 1$, where $c := \frac{2d}{3} + \frac{1}{54}$.

Assume to the contrary that this is not the case. Then C_{int} has at least $\lceil \frac{n}{2} \rceil - cn$ interior faces that are not interior faces of H_{int}. We partition these faces into four subsets F_0, F_1, F_2 and $F_{>3}$, according to whether their boundary is a 0-, 1-, 2-triangle of C or has more than three edges. The weight sum of the vertices of D that are not in the interior of H is $\lceil \frac{n}{2} \rceil - dn$. Thus, $\sum_{i=0}^{2} |F_i| + 2|F_{>3}| \leq \lceil \frac{n}{2} \rceil - dn$. Moreover, since $\sum_{i=0}^{2} |F_i| + |F_{>3}| \geq \lceil \frac{n}{2} \rceil - cn$, subtracting the former inequality from two times the latter gives $\sum_{i=0}^{2} |F_i| \geq \lceil \frac{n}{2} \rceil - 2cn + dn$.

Note that if we root the tree D at a vertex that corresponds to an arbitrary interior face of H, then every face in $F_0 \cup F_1$ has at least one descendant in $F_{>3}$. Thus, we have $|F_0| + |F_1| \leq |F_{>3}|$. Hence, $\lceil n/2 \rceil - dn \geq \sum_{i=0}^{2} |F_i| + 2|F_{>3}| \geq 3\sum_{i=0}^{2} |F_i| - 2|F_2| \geq 3(\lceil n/2 \rceil - 2cn + dn) - 2|F_2|$. This implies $|F_2| \geq \lceil n/2 \rceil - 3cn + 2dn$, which is a lower bound on the number of 2-triangles of C in C_{int}.

Let $\{u, v, w\}$ be a 2-triangle of C in C_{int} such that $uv, vw \in E(C)$. Then v is a vertex of either a 0- or 1-triangle of C in C_{ext} and neither uv nor vw is contained in a 1-triangle of C. By the above lower bound on the number of 2-triangles, C_{ext} contains at most $n - 2(\lceil n/2 \rceil - 3cn + 2dn) \leq (6c - 4d)n$ 1-triangles of C and at least $\frac{1}{3}((\lceil n/2 \rceil - 3cn + 2dn) - (6c - 4d)n) \geq (1/6 - 3c + 2d)n$ 0-triangles of C. Now consider the subgraph of G that is induced by the edges of $E(C)$ that are contained in some 2-triangle of C contained in C_{int}; clearly, this subgraph is a union of l vertex-disjoint paths. No such path may contain two vertices from a 0-triangle T of C in C_{ext}, as otherwise G would have a 3-separator (consisting of one vertex of T and neighbors of two other vertices of T). By this observation, it can be readily shown that l is at least the number of 0-triangles of C in C_{ext} plus two (by induction on the number of 0-triangles). Since C_{ext} contains at least $(1/6 - 3c + 2d)n$ 0-triangles of C, we have

$$l \geq (1/6 - 3c + 2d)n + 2.$$

On the other hand, the edges from $E(C)$ that are not contained in any 2-triangle of C in C_{int} also induce l vertex-disjoint paths. We deduce that $l \leq n - 2(\lceil n/2 \rceil - 3cn + 2dn) \leq (6c - 4d)n$. This contradicts the previous bound on l, as $1/6 - 3c + 2d = 6c - 4d$. Hence, $f > cn + 1$, which proves the lemma. □

Theorem 7. *Every contraction-critically 4-connected planar graph G on n vertices contains a cycle of length k for every $k \in \{\lfloor \frac{n}{2} \rfloor - \lceil \frac{n}{108} \rceil, \ldots, \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{36} \rfloor\}$.*

Proof. Let C be a hamiltonian cycle of G. Since $|E(G)| = 2n$, C_{int} or C_{ext} contains at least $n/2$ chords of C, say without loss of generality C_{int}. By Corollary 3, C_{int} contains a k-cycle for every $k \in \{\lfloor \frac{n}{2} \rfloor, \ldots, \lceil \frac{n}{2} \rceil + 3\}$. Let H be a $\lfloor \frac{n}{2} \rfloor$-cycle in C_{int}. Since $n \geq 9$ (as G is not the octahedral graph and n is a multiple of three), H has length at least four. Then applying Lemma 6 with $d := 0$ shows that there are more than $n/108$ 2-triangles of H. Short-cutting H by these 2-triangles gives us a k-cycle in G for every $k \in \{\lfloor \frac{n}{2} \rfloor - \lceil \frac{n}{108} \rceil, \ldots, \lfloor \frac{n}{2} \rfloor\}$.

We may now assume $n \geq 108$, as otherwise $\lceil \frac{n}{2} \rceil + 3 > \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{36} \rfloor$. Let $k \in \{\lceil \frac{n}{2} \rceil + 3, \ldots, \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{36} \rfloor\} \subset \{\lceil \frac{n}{2} \rceil + 3, \ldots, \frac{3n+7}{4}\}$. We have that $k = \lfloor n/2 \rfloor + dn$ for some $0 \leq d \leq 1/36$. By Corollary 3, C_{int} has a cycle H of length $k' \in \{k, \ldots, 2k - \lceil \frac{n}{2} \rceil - 3\}$, so that $k' = \lfloor n/2 \rfloor + d'n$ for some $d \leq d' \leq 2d$. According to Lemma 6, there are more than $(d'/3 + 1/108)n$ 2-triangles of H. Since $d'/2 \leq d \leq \frac{1}{36}$, we have $k' - k = (d' - d)n \leq (d'/3 + 1/108)n$, so that a k-cycle can be obtained from H by short-cutting 2-triangles of H. This completes the proof. □

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