Longest Cycles in Cyclically 4-Edge-Connected Cubic Planar Graphs*

On-Hei Solomon Lo
Institute of Mathematics
TU Ilmenau, Germany

Jens M. Schmidt
Institute of Mathematics
TU Ilmenau, Germany

Abstract

Grünbaum and Malkevitch proved in 1976 that the shortness coefficient of cyclically 4-edge-connected cubic planar graphs is at most $\frac{76}{77}$. We prove that it is at most $\frac{359}{366} (< \frac{52}{53})$.

1 Introduction and Notation

We consider only simple and finite graphs, unless otherwise specified. Given a graph G, we denote by n the number $|V(G)|$ of vertices of G, we also call G an *n-vertex* graph. A *k-separator* (*k-edge cut*) S of G is a subset of k vertices (edges) such that $G - S$ is disconnected. A graph G is 3-*connected* if it has no 2-separator, and it is *essentially* 4-*connected* if in addition, for every 3-separator S of G, $G - S$ has a component that is a single vertex. A *fragment* of G is a subgraph along with some *half-edges* of G. If a fragment has k half-edges, we call it a *k-leg fragment* (see Figure 1 for an example of a 4-leg fragment). For vertices $x, y \in V(G)$, an *x-y-path* is a path from x to y; this notation is also extended to objects other than vertices, for instance, edges and half-edges.

The *circumference* $\mathrm{circ}(G)$ of a graph G denotes the length of a longest cycle in G. For infinite graph classes, Grünbaum and Walther [7] proposed the following measure: The *shortness coefficient* $\rho(\mathcal{F})$ of an infinite graph class $\mathcal{F}$ is $\liminf_{G \in \mathcal{F}, n \to \infty} \frac{\mathrm{circ}(G)}{n}$.

*This research is supported by the grant SCHM 3186/1-1 (270450205) from the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation), by DAAD (as part of BMBF, Germany) and by the Ministry of Education Science, Research and Sport of the Slovak Republic within the project 57320575.

A famous result by Tutte [10] states that every 4-connected planar graph is Hamiltonian; hence, 4-connected planar graphs have shortness coefficient 1. However, infinitely many planar graphs of connectivity 3 and with no spanning cycle exist. Indeed, Moon and Moser [9] showed that there are infinitely many 3-connected planar graphs with circumference at most $O(n^{\log_3 2})$, while Chen and Yu [3] showed that every 3-connected planar graph has circumference at least $\Omega(n^{\log_3 2})$; hence, the shortness coefficient of 3-connected planar graphs is 0. Restricted to cubic graphs, the shortness coefficient is still 0 [2], while Liu et al. [8] showed that every 3-connected cubic planar graph has circumference at least $\Omega(n^{0.8})$.

In this paper, we are interested in cyclically 4-edge-connected cubic planar graphs. A graph G is *cyclically* 4*-edge-connected* if it is 3-connected and, for every 3-edge cut S of G, at most one component of $G - S$ contains a cycle. Note that for 3-connected cubic graphs different from the *prism graph* $K_2 \Box K_3$ (where $\Box$ is the cartesian graph product), *cyclic 4-edge-connectivity* and *essential 4-connectivity* coincide. Let $\mathcal{C}4$ be the class of cyclically 4-edge-connected cubic planar graphs. It is known that the shortness coefficient of $\mathcal{C}4$ lies strictly between 0 or 1. Aldred et al. [1] showed that the smallest non-Hamiltonian graphs in $\mathcal{C}4$ have 42 vertices and that there are exactly three such graphs, including the Grinberg graph [5] and the Faulkner-Younger graph [4] on 42 vertices. It is however not known whether $\rho(\mathcal{C}4) \leq \frac{41}{42}$ holds.

Grünbaum and Malkevitch [6] proved in 1976 that every graph in $\mathcal{C}4$ has circumference at least $\frac{3}{4}n$, and invoking the theory of Tutte paths gives the slightly improved lower bound $\frac{3}{4}n + 1$ (see also [12]). By constructing a graph H from fragments of the 42-vertex Grinberg graph and replacing every vertex of a 4-regular 4-connected planar graph with a copy of H, they also showed that there are infinitely many graphs in $\mathcal{C}4$ with circumference at most $\frac{76}{77}n$, which implies $\rho(\mathcal{C}4) \leq \frac{76}{77}$. This is the best known upper bound so far.[1] We improve this to $\rho(\mathcal{C}4) \leq \frac{359}{366} (< \frac{52}{53})$. Our method is similar to the one by Grünbaum and Malkevitch, but we use fragments of the 42-vertex Faulkner-Younger graph (instead of the Grinberg graph) that are comparably smaller and replace the vertices of a more restricted class of 4-regular planar multigraphs.

[1]During the proof of this paper, the authors were made aware of a comment by Zaks [11, p. 97], which states that one can "easily deduce" $\rho(\mathcal{C}4) \leq \frac{76}{78}$ from the infinite graph family M_k described in the paper of Faulkner and Younger [4]. However, the copies of B_3 that are used in M_k have only penalty 1 and not penalty 2 (see Section 2 for the definition of *penalty*), and hence, using only the family M_k, one can deduce only $\rho(\mathcal{C}4) \leq \frac{77}{78}$.

2 An Upper Bound for the Shortness Coefficient

Grünbaum and Malkevitch [6] extracted a 38-vertex fragment H from the 42-vertex Grinberg graph [5] by deleting the vertices of its only 4-face, and then constructed a 154-vertex 4-leg fragment F from 4 copies of H by adding two vertices. They showed that if a graph G contains a copy of F and a cycle C that is not fully contained in F, then C contains at most 152 vertices of F. This implies $\rho(\mathcal{C}4) \leq \frac{152}{154} = \frac{76}{77}$, as then for any 4-regular 4-connected planar graph (and there are infinitely many such graphs), we can replace every vertex with a copy of F, which gives a graph in $\mathcal{C}4$. We will use a similar method, but use a smaller 122-vertex 4-leg fragment than F (this smaller fragment implies already $\rho(\mathcal{C}4) \leq \frac{60}{61}$) and replace the vertices of graphs from a more restricted graph class to prove the following theorem:

Theorem 1. *For the class $\mathcal{C}4$ of cyclically 4-edge-connected cubic planar graphs, $\rho(\mathcal{C}4) \leq \frac{359}{366}$.*

In contrast to Grünbaum and Malkevitch, who used a big fragment of the Grinberg graph, we will use the following 4-leg fragment H_1 of the Faulkner-Younger graph (see Figure 1). This 4-leg fragment has 18 vertices and can also be obtained from the dodecahedron graph by deleting the two endvertices of any edge (leaving four half-edges); let l^- and l^+ be the half-edges at the left lower and upper corners and let r^- and r^+ be the half-edges at the right lower and upper corners. Faulkner and Younger noted the following property of H_1.

Lemma 2 ([4, Lemmas 2.1 and 2.2])**.** *Let G be a graph containing H_1 and let C be a cycle of G that contains $V(H_1)$. Then $C \cap H_1$ contains either a l^--r^--path (or by symmetry a l^+-r^+-path) or a l^--r^+-path (or by symmetry a l^+-r^--path).*

In the first case of Lemma 2, $C \cap H_1$ may contain both an l^--r^--path and an l^+-r^+-path, but not more, as H_1 has only four legs. The path of the latter case of Lemma 2 however is the only one in the intersection, as H_1 is plane. Using H_1, we construct the following 60-vertex 4-leg fragment H_2, which consists of three copies of H_1 (which we call A, B and D from top to bottom) and 6 more vertices (see Figure 2a).

Now let G be a graph containing H_2 and let C be a cycle of G. We say that $C \cap H_2$ has *penalty* k if C misses at least k vertices of H_2. Then $C \cap H_2$ is either empty, C, or consists of one or of two paths. The first two cases will eventually only be beneficial. In the last case, it is again not possible to have one l^--r^+-path and one l^+-r^--path simultaneously by planarity. As H_2 is

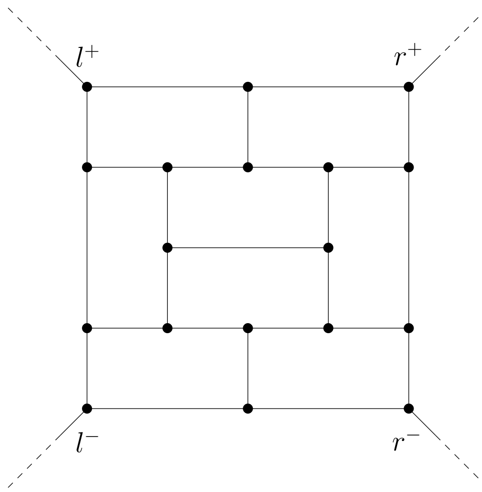
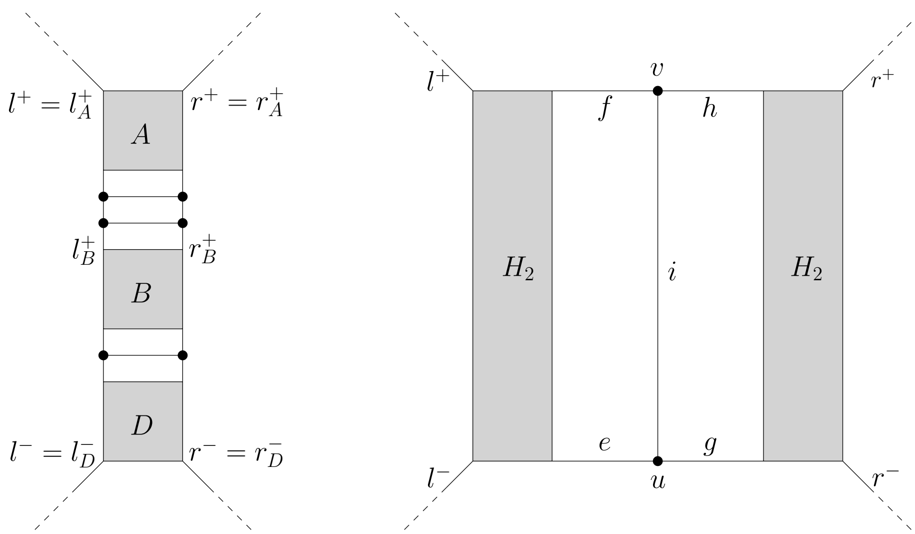


Figure 1: The 4-leg fragment H_1.



(a) The fragment H_2.

(b) The fragment H_3.

Figure 2: Bigger 4-leg fragments.

mirror symmetric along a vertical line, it suffices to consider the following cases.

Lemma 3. *If $l^- \in C \cap H_2$, then $C \cap H_2$ consists of one of the following:*

(i) one l^--l^+-path such that $C \cap H_2$ has penalty 0,

(ii) *one* l^-*-*r^+*-path such that* $C \cap H_2$ *has penalty* 1*,*
(iii) *one* l^-*-*r^-*-path such that* $C \cap H_2$ *has penalty* 2*,*
(iv) *one* l^-*-*l^+*-path and one* r^-*-*r^+*-path such that* $C \cap H_2$ *has penalty* 3*, or*
(v) *one* l^-*-*r^-*-path and one* l^+*-*r^+*-path such that* $C \cap H_2$ *has penalty* 2*.*

If $C \cap H_2$ *consists of (vi) one* l^+*-*r^+*-path, then* $C \cap H_2$ *has penalty* 2*.*

Proof. In Case (i), there is nothing to show, as the penalty is 0. Note that we cannot demand a higher penalty in this case, as the l^--l^+-path that uses only diagonally opposite half-edges of A, B and D shows.

In Case (ii), the path either contains an l_F^--l_F^+-path (or an r_F^--r_F^+-path) for some $F \in \{A, B, D\}$ or misses at least one of the six vertices that are in no copy of H_1. By Lemma 2, $C \cap H_2$ has penalty 1.

In Case (iii), if the path is an l^--r^--path, we may assume that the path contains a vertex of A. Then each of B and D contains two top-down paths in its intersection with C. By Lemma 2, this gives a total penalty of 2. If the path is an l^+-r^+-path, an analogous statement settles that $C \cap H_2$ has penalty 2. In Case (vi), $C \cap H_2$ has penalty 2 for the same reason.

In Case (iv), the planarity of H_1 implies that each of A, B and D contains two top-down paths, which amounts to penalty 3 in total by Lemma 2.

In Case (v), if the l^--r^--path P contains a vertex of the four vertices between A and B, then the argument of Case (iii) gives penalty 2. Similarly, it gives penalty 2 if the l^+-r^+-path Q contains a vertex of the two vertices between B and D. In the remaining case, if neither P nor Q intersects B, the penalty is clearly at least $|V(B)| \geq 2$. Otherwise at least one of P and Q, say without loss of generality P, contains a vertex of B. By Lemma 2, D gives penalty 1. Now we have either Q contains all four vertices between A and B, then A gives penalty 1, or some of the four vertices contained neither in P nor Q. In any case, we have penalty 2 in total. □

We obtain the 4-leg fragment H_3 from two copies of H_2 by adding two new vertices u and v as shown in Figure 2b. Hence, H_3 has 122 vertices. Using the same notation and assumptions as for H_2, we have the following lemma.

Lemma 4. *If* $l^- \in C \cap H_3$*, then* $C \cap H_3$ *consists of one of the following:*
(i) *one* l^-*-*l^+*-path such that* $C \cap H_3$ *has penalty* 2*,*
(ii) *one* l^-*-*r^+*-path such that* $C \cap H_3$ *has penalty* 2*,*
(iii) *one* l^-*-*r^-*-path such that* $C \cap H_3$ *has penalty* 3*,*
(iv) *one* l^-*-*l^+*-path and one* r^-*-*r^+*-path such that* $C \cap H_3$ *has penalty* 2*, or*
(v) *one* l^-*-*r^-*-path and one* l^+*-*r^+*-path such that* $C \cap H_3$ *has penalty* 4*.*

If $C \cap H_3$ *consists of (vi) one* l^+*-*r^+*-path, then* $C \cap H_3$ *has penalty* 3*.*

Proof. Let e, f, g, h and i be the edges of H_3 as indicated in Figure 2b. In Case (i), if the edges e and f are not in C, we have penalty $|V(H_2)|+2 \geq 2$, so assume they are in C. Then the left copy of H_2 is in Case (v) of Lemma 3, which gives penalty 2.

In Cases (ii) and (iii), the respective path P must contain exactly one of the edges e and f and exactly one of g and h. If $e \in P$, we have penalty 2 by Case (iii) of Lemma 3, and if $f \in P$, we have penalty 1 by Case (ii) of Lemma 3. Using the symmetric arguments for g and h and r^- or r^+, we thus have total penalty 2 in Case (ii) (C misses exactly two vertices of H_3 only if $i \in P$) and total penalty 3 in Case (iii) (as either one of the vertices u and v is missing or we have either an l^--e-path or an g-r^--path, all of which adds 1 to the penalty). In Case (vi), $C \cap H_2$ has penalty 3 for the same reason.

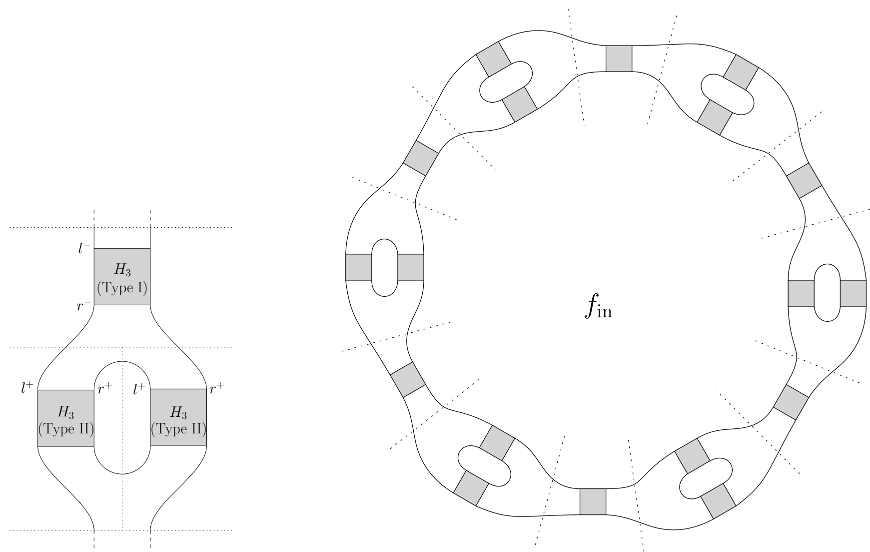
In Case (iv), either one path implies penalty 2 due to Case (v) of Lemma 3 or the vertices of each of the two paths P and Q are the vertices of a copy of H_2 (each implying penalty 0 by Case (i) of Lemma 3). Then the penalty is 2 in total, as the latter case misses the end vertices of i.

In Case (v), the l^--r^--path contains the edges e and g and the l^+-r^+-path contains the edges f and h by planarity of H_3. Hence both copies of H_2 give penalty 2 by Case (v) of Lemma 3, which implies total penalty 4. $\square$

Now consider any 4-regular 4-connected planar graph G' and replace each vertex along with its four incident half-edges with the 4-leg fragment H_3. The graph G on n vertices obtained in this way is cubic, cyclically 4-edge-connected and planar and thus in $\mathcal{C}4$. Let C be a longest cycle of G. If C is contained in some copy of H_3, we have $\operatorname{circ}(G) \leq \frac{1}{2}n$, as $|V(G')| \geq 2$. Otherwise, every copy of H_3 in G has penalty at least 2 due to Lemma 4, and hence $\operatorname{circ}(G) \leq \frac{120}{122}n = \frac{60}{61}n$, since there are infinitely many 4-regular 4-connected planar graphs.

This implies already $\rho(\mathcal{C}4) \leq \frac{60}{61}$. In the following, we will further improve this bound by replacing vertices of some more specific graphs, such that any cycle in the constructed graphs must encounter many cases of high penalty, namely Cases (iii), (v) and (vi) of Lemma 4.

Let H_4 be the 4-leg fragment shown in Figure 3 that contains two copies of H_3 in the same orientation as in Figure 2 (we say that these copies are of *Type II*) and one copy of H_3 that is rotated by 90 degrees (we say that this copy is of *Type I*). Let G_k be the graph obtained from linking k copies of H_4 in a cyclic way, as shown in Figure 3b. It is not difficult to check that G_k is in $\mathcal{C}4$. Let C be a longest cycle of G_k. Then C divides the plane into two open sets; let $\operatorname{in}(C)$ be the bounded (inner) open set and let $\operatorname{out}(C)$ be



(a) The fragment H_4 (curves depict edges).

(b) The graph G_6 (the grey subgraphs depict copies of H_3).

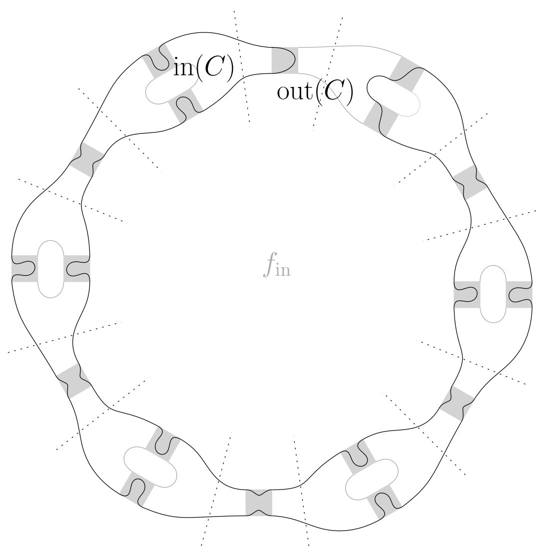
Figure 3: Construction of the graph G_6.

the unbounded (outer) open set.

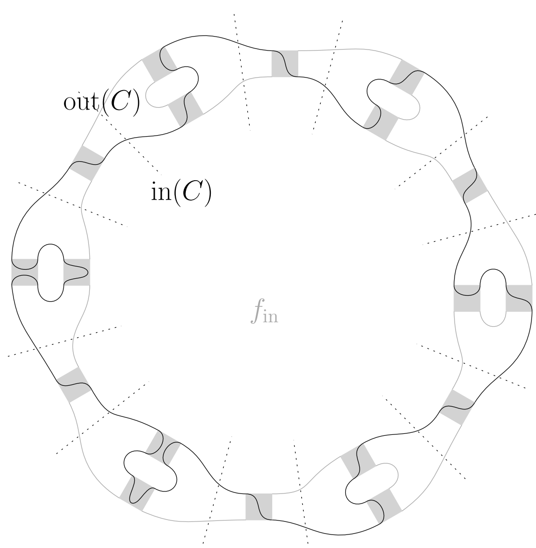
If the face f_{in} of G_k (see Figure 3b) intersects out(C) (see Figure 4a), then every edge pair that is cut by a dotted line segment of Figure 4a has the property that either both edges are in C or none of them is in C. By the maximality of C, the latter case can happen at most once. Therefore, up to this one exceptional edge pair, every Type I-copy of H_3 has one l^--r^--path and one l^+-r^+-path when intersecting with the cycle C, which gives penalty 4 due to Case (v) of Lemma 4. Since every Type II-copy of H_3 has penalty at least 2 by Lemma 4, every copy of H_4 has penalty 8, except possibly one which has penalty 6 (when its edges cut by dotted line segments are not all contained in the cycle). Hence, for $k \geq 2$, every copy of H_4 has penalty at least 7 on average.

If f_{in} does not intersect out(C), it intersects in(C) (see Figure 4b). Then C contains exactly one edge from every edge pair that is cut by a dotted line segment of Figure 4b. Since C has maximal length, both Type II-fragments in every copy of H_4 of G_k intersect C. For the intersection of C with these two Type II-copies, we distinguish the following two cases:

(a) *C enters the first Type II-copy from outside, proceeds to the second*



(a) f_{in} intersects out(C).



(b) f_{in} intersects in(C).

Figure 4: Two cases of a cycle C in G_6.

Type II-copy, and then returns to the first copy before it leaves the intersection.

Then the first copy is in Case (v) of Lemma 4, and the second is in

Case (i) of Lemma 4.

(b) *C enters the first Type II-copy from outside, proceeds to the second Type II-copy, and does not return to the first copy.*
Then one copy is in Case (iii) or (vi) of Lemma 4 and the other in Case (ii) of Lemma 4.

Therefore, for both Cases (a) and (b), at least one H_3-copy has penalty 3, and thus any H_4-copy has penalty 7 in total.

Since every copy of H_4 has penalty 7, the shortness coefficient $\rho(\mathcal{C}4)$ is therefore at most $\frac{3\cdot 122-7}{3\cdot 122} = \frac{359}{366} < \frac{52}{53}$. This completes the proof of Theorem 1.

References

[1] R. E. L. Aldred, S. Bau, D. A. Holton, and B. D. McKay. Non-hamiltonian 3-connected cubic planar graphs. *SIAM J. Discrete Math.*, 13(1):25–32, 2000.

[2] J. A. Bondy and M. Simonovits. Longest cycles in 3-connected 3-regular graphs. *Canad. J. Math.*, 32:987–992, 1980.

[3] G. Chen and X. Yu. Long cycles in 3-connected graphs. *J. Combin. Theory Ser. B*, 86(1):80–99, 2002.

[4] G. B. Faulkner and D. H. Younger. Non-Hamiltonian cubic planar maps. *Discrete Math.*, 7(1-2):67–74, 1974.

[5] E. J. Grinberg. Plane homogeneous graphs of degree three without Hamiltonian circuits. *Latvian Math. Yearbook*, 4:51–58, 1968. (in Russian, a translated version is available at http://www.ltn.lv/~dainize/MathPages/Grinberg.eng.article.pdf).

[6] B. Grünbaum and J. Malkevitch. Pairs of edge-disjoint Hamiltonian circuits. *Aequationes Math.*, 14(1):191–196, 1976.

[7] B. Grünbaum and H. Walther. Shortness exponents of families of graphs. *J. Combin. Theory Ser. A*, 14(3):364–385, 1973.

[8] Q. Liu, X. Yu, and Z. Zhang. Circumference of 3-connected cubic graphs. *J. Combin. Theory Ser. B*, 128:134–159, 2018.

[9] J. W. Moon and L. Moser. Simple paths on polyhedra. *Pacific J. Math.*, 13(2):629–631, 1963.

[10] W. T. Tutte. A theorem on planar graphs. *Trans. Amer. Math. Soc.*, 82(1):99–116, 1956.

[11] J. Zaks. Shortness coefficient of cyclically 5-connected cubic planar graphs. *Aequationes Math.*, 25:97–102, 1982.

[12] C.-Q. Zhang. Longest cycles and their chords. *J. Graph Theory*, 11(4):521–529, 1987.