Contractible Edges in Longest Cycles

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Abstract

This paper investigates the number of contractible edges in a longest cycle C of a k-connected graph ($k \geq 3$) that is triangle-free or has minimum degree at least $\frac{3}{2}k - 1$. We prove that, except for two graphs, C contains at least $\min\{|E(C)|, 6\}$ contractible edges. For triangle-free 3-connected graphs, we show that C contains at least $\min\{|E(C)|, 7\}$ contractible edges, and characterize all graphs having a longest cycle containing exactly six/seven contractible edges. Both results are tight. Lastly, we prove that every longest cycle C of a 3-connected graph of girth at least 5 contains at least $\frac{|E(C)|}{12}$ contractible edges.

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1 Introduction

The study of contractible edges began with Tutte [25], who proved that every 3-connected graph non-isomorphic to K_4 contains a contractible edge. Since then, a lot of research has been done on the existence and distribution of contractible edges in k-connected graphs. The surveys by Kriesell [18] and Ando [4] collect and summarize many results in this area.

One interesting question to ask is what classes of subgraphs contain a contractible edge and how many. For every 3-connected graph other than K_4 and the prism $K_2 \times K_3$, Dean et al. [7] proved that every longest cycle contains at least three contractible edges. Later, Ellingham et al. [11] proved that for any non-hamiltonian 3-connected graph, every longest cycle contains at least six contractible edges. Aldred et al. [1] characterized all 3-connected graphs that have a longest cycle containing exactly three contractible edges. Fujita [13, 16] classified all 3-connected graphs that have a longest cycle containing precisely four contractible edges. Fujita and Kotani [17] classified all 3-connected graphs of order at least 16 that have a longest cycle containing precisely five contractible edges. For any 3-connected graph of order at least five, Fujita [14] proved that there exists a longest cycle C such that C contains at least $\frac{|V(C)|+9}{8}$ contractible edges, and later [15] improved the lower bound to $\frac{|V(C)|+7}{7}$.

Besides longest cycles, maximum matchings in 3-connected graphs non-isomorphic to K_4 were shown to contain a contractible edge by Aldred et al. [2]. They [3] also characterized all 3-connected graphs with a maximum matching that contains precisely one contractible edge. Elmasry et al. [12] proved that every depth-first search (DFS) tree in a 3-connected graph non-isomorphic to K_4 contains a contractible edge. Recently, Kriesell and Schmidt [20] made an improvement, and proved that every DFS tree of a 3-connected graph non-isomorphic to K_4, the prism or the prism plus an edge has two contractible edges.

Note that all the above work focuses on 3-connected graphs, as k-connected graphs may not contain any contractible edge for every $k \geq 4$. By imposing extra conditions, Thomassen [24] proved that every triangle-free k-connected graph contains a contractible edge (see also Egawa et al. [10]), while Egawa [9] proved the same for k-connected graphs that have minimum degree at least $\lfloor 5n/4 \rfloor$. For k-connected graphs that are triangle-free or have minimum degree at least $\frac{3}{2}k-1$, Kriesell and Schmidt [20] proved that except for the graphs K_{k+1} where $k = 1, 2$, every spanning tree has two contractible edges. For every k-connected graph ($k \geq 4$) of minimum degree at least $\frac{3}{2}(k-1)$, they showed that every DFS tree has two contractible edges.

Inspired by these results, in this paper, we initiate the investigation of the number of contractible edges in longest cycles of k-connected graphs ($k \geq 3$) that are triangle-free or have minimum degree at least $\frac{3}{2}k-1$. More specifically, we study the largest value $f(k)$ such that every longest cycle C contains at least $\min\{|E(C)|, f(k)\}$ contractible edges. Section 2 introduces the necessary terminology and basic tools used throughout the paper. Section 3 gives lower bounds for $f(k)$. We prove that, except for two graphs, C contains at least $\min\{|E(C)|, 6\}$ contractible edges. In Section 4, by constructing specific examples, we find upper bounds for $f(k)$. Section 5 deals with triangle-free 3-connected graphs exclusively. We show that C contains at least $\min\{|E(C)|, 7\}$ contractible edges, and characterize all graphs having a longest cycle containing exactly six/seven contractible edges. In Section 6, we consider 3-connected graphs of girth at least 5 and prove that every longest cycle C in these graphs contains at least $\frac{|E(C)|}{12}$ contractible edges, which is a first dependency result on $|E(C)|$ applicable to all longest cycles. In Section 7, all the results are summarized and open problems are raised.

2 Preliminaries

All graphs throughout the paper are assumed to be finite, simple and undirected. For terminology not defined here, we refer to [8] or [5]. For a graph G, a set $T \subseteq V(G)$ is a $|T|$-*separator* of G if $G - T$ is disconnected. For $k \geq 1$, a noncomplete graph G is called k-*connected* if $|V(G)| > k$ and G does not contain a $(k-1)$-separator. Let x be a vertex in G and S be a subset of $V(G)$ such that $x \notin S$ and $|S| \geq l$. An x-S l-*fan* is the union of l distinct x-S paths that pairwise intersect exactly at x. By Menger's theorem, an x-S k-fan always exists. Let $\kappa(G)$ denote the *connectivity* of G, that is, the largest k such that G is k-connected. Define $\mathfrak{T}(G)$ to be the set of all $\kappa(G)$-separators of G, or simply write $\mathfrak{T}$ if it is clear from the context. For an edge e, let $V(e)$ be the set of endvertices of e. An edge e of a k-connected graph G is called k-*contractible* if the graph G/e obtained from G by *contracting* e, that is, identifying its endvertices, and removing loops and multiple edges, is k-connected. No edge in K_{k+1} is k-contractible, but all edges in K_ℓ, $\ell \geq k+2$, are. It is well-known and straightforward to check that an edge e of a noncomplete k-connected graph G is not k-contractible if and only if $\kappa(G) = k$ and $V(e) \subseteq T$ for some $T \in \mathfrak{T}$. In the following, we will write k-contractible as *contractible* if the context is clear. For two disjoint vertex subsets X and Y of a graph G, let $E_G(X)$ be the set of edges that X induces in G and let $E_G(X,Y) := \{xy \in E(G) : x \in X \text{ and } y \in Y\}$. For the k-connected graphs that are either triangle-free or have minimum degree at least $\frac{3}{2}k - 1$, define $f(k)$ to be the largest value such that every longest cycle C in these graphs contains at least $\min\{|E(C)|, f(k)\}$ contractible edges.

For $T \in \mathfrak{T}$, a T-*fragment* is the union of vertex sets of at least one but not all components of $G - T$. A *fragment* A of G is a T-fragment for some $T \in \mathfrak{T}$. For any fragment A, we define $T_A := N_G(A)$ where $N_G(A)$ is the set of neighbors of vertices in A that lie outside A. Obviously, A is a T_A-fragment. Define $\overline{A} := (V(G) - T_A) - A$, which is another T_A-fragment. Given a graph G and a non-empty set $\mathfrak{S} \subseteq 2^{V(G)}$ (where $2^{V(G)}$ is the power set of $V(G)$), an $\mathfrak{S}$-*fragment of* G is a T-fragment for some $T \in \mathfrak{T}$ such that there is an $S \in \mathfrak{S}$ satisfying $S \subseteq T$. An $\mathfrak{S}$-*end* is an inclusion-wise minimal $\mathfrak{S}$-fragment, and an $\mathfrak{S}$-*atom* is an $\mathfrak{S}$-fragment of minimum size.

Not every k-connected graph contains a contractible edge, but every k-connected graph that is triangle-free or has large minimum degree does. We will need the following well-known lemmas. The first is a standard argument in k-connectivity and will therefore often be used without explicit reference throughout this paper; the other two ensure that all fragments are large.

Lemma 1 (Fundamental Lemma [22, Lemma 1])**.** *Let B and F be fragments such that $B \cap F \neq \emptyset$. Then $|B \cap T_F| \geq |\overline{F} \cap T_B|$. If the equality holds, $B \cap F$ is an $(B \cap T_F) \cup (T_B \cap T_F) \cup (F \cap T_B)$-fragment.*

Lemma 2 ([24])**.** *Let G be a k-connected triangle-free graph and S be a k-separator that contains $V(e)$ for an edge e. Then, for every fragment F of $G - S$, $|F| \geq k$.*

Lemma 3 ([21])**.** *Let G be a graph of connectivity k with minimum degree at least $\frac{3}{2}k-1$. Then every fragment F of G satisfies $|F| \geq \frac{k}{2}$.*

3 Lower Bounds for $f(k)$

We first prove that, for a subgraph H of G and $\mathfrak{S} := \{V(e) : e \in E(H)\}$, all edges in H intersecting an $\mathfrak{S}$-end are contractible if all $\mathfrak{S}$-fragments are large. This is a sharper version of Lemma 2 in [6].

Lemma 4. *Let H be a subgraph of a k-connected graph G, $\mathfrak{S} := \{V(e) : e \in E(H)\}$ and suppose that every $\mathfrak{S}$-fragment has at least $\frac{k}{2}$ vertices. Let B be an $\mathfrak{S}$-end and e be an edge of H such that $|V(e) \cap B| \geq 1$. Suppose e is non-contractible and let T be any k-separator containing $V(e)$. Then $B \subseteq T$, $|B| = |\overline{B} \cap T| = \frac{k}{2}$, $T \cap T_B = \emptyset$ and $|V(e) \cap B| = 2$.*

Proof. Let F be a T-fragment that is also an $\mathfrak{S}$-fragment. Note that $B \cap T \cap V(e) \neq \emptyset$ and $V(e) \subseteq (B \cap T) \cup (T \cap T_B)$.

Case 1: $F \cap B \neq \emptyset$ and $\overline{F} \cap B \neq \emptyset$. Since $F \cap B$ and $\overline{F} \cap B$ are not $\mathfrak{S}$-fragments, $|F \cap T_B| > |\overline{B} \cap T|$ and $|\overline{F} \cap T_B| > |\overline{B} \cap T|$ by Lemma 1. Then $F \cap \overline{B} = \emptyset = \overline{F} \cap \overline{B}$. Hence, $|\overline{B} \cap T| = |\overline{B}| \geq \frac{k}{2}$. But now, $|F \cap T_B| > \frac{k}{2}$ and $|\overline{F} \cap T_B| > \frac{k}{2}$, contradicting $|T_B| = k$.

Case 2: $F \cap B \neq \emptyset$ and $\overline{F} \cap B = \emptyset$ (or vice versa by symmetry of F and $\overline{F}$). Since $F \cap B$ is not an $\mathfrak{S}$-fragment, $|F \cap T_B| > |\overline{B} \cap T|$ and $|B \cap T| > |\overline{F} \cap T_B|$ by Lemma 1. Then $\overline{F} \cap \overline{B} = \emptyset$ and $|\overline{F} \cap T_B| = |\overline{F}| \geq \frac{k}{2}$. This implies $|B \cap T| > \frac{k}{2}$ and $|\overline{B} \cap T| < \frac{k}{2}$. Hence, $F \cap \overline{B} \neq \emptyset$. By Lemma 1, $|\overline{B} \cap T| \geq |\overline{F} \cap T_B|$, which is impossible.

Case 3: $F \cap B = \emptyset$ and $\overline{F} \cap B = \emptyset$. Then $B \subseteq T$ and $|B \cap T| = |B| \geq \frac{k}{2}$ and $|\overline{B} \cap T| \leq \frac{k}{2}$. If $|\overline{B} \cap T| < \frac{k}{2}$, then $\overline{B} \not\subseteq T$; assume without loss of generality that $F \cap \overline{B} \neq \emptyset$. Then by Lemma 1, $|\overline{F} \cap T_B| \leq |\overline{B} \cap T| < \frac{k}{2}$, which implies $\overline{F} \cap \overline{B} \neq \emptyset$. By Lemma 1, $|\overline{F} \cap T_B| \geq |B \cap T|$, which is impossible. Hence, $|\overline{B} \cap T| = \frac{k}{2} = |B \cap T|$. We have $T \cap T_B = \emptyset$, $|V(e) \cap B| = 2$ and $|B| = \frac{k}{2}$. □

The next lemma is a slightly stronger version of the result of Dean, Hemminger and Ota [7] that every longest cycle C intersects every $\mathfrak{S}$-fragment, where $\mathfrak{S} := \{V(e) : e \in E(C)\}$.

Lemma 5. *Let C be a longest cycle in a graph G satisfying $\kappa(G) \geq 3$. Suppose C contains a non-contractible edge. Let $\mathfrak{S} := \{V(e) : e \in E(C)\}$ and F be any $\mathfrak{S}$-fragment. Then $|V(C) \cap F| \geq 1$ and $|E(C) \cap E_G(F, T_F)| \geq 2$. If $|F| > 1$, then $|V(C) \cap F| > 1$.*

Proof. Let uv be any edge of $C \cap G[T_F]$. Assume to the contrary that F does not intersect C. Let u' be a neighbor of u in F, v' be a neighbor of v in the same component of F containing u', and P be a u'-v' path in F. Then we can replace uv by $uu'Pv'v$ to obtain a longer cycle, which contradicts that C is a longest cycle. Therefore, $|V(C) \cap F| \geq 1$ and $|E(C) \cap E_G(F, T_F)| \geq 2$.

For the second statement, assume to the contrary that $|V(C) \cap F| = 1$. Let $\{z\} := V(C) \cap F$ and x, y be the two vertices adjacent to z in C. Note that $x, y \in T_F$ and $\{x, y\} \neq \{u, v\}$. Let a be a vertex in F other than z. Consider an a-$(T_F \cup \{z\})$ $\kappa(G)$-fan H.

Suppose first that $z \notin H$. Let P_u be the a-u path in H and P_v be the a-v path in H. Then we can replace uv by uP_uaP_vv to obtain a longer cycle, which is a contradiction. Suppose now $z \in H$ and assume without loss of generality that $x \in H$. Let P_x be the a-x path in H and P_z be the a-z path in H. Then we can replace xz by xP_xaP_zz to obtain a longer cycle, which is a contradiction. $\square$

The following lemma gives us precise structural information about a fragment when its intersection with a longest cycle is small.

Lemma 6. *Let C be a longest cycle in a graph G satisfying $\kappa(G) \geq 3$. Suppose C contains a non-contractible edge. Let $\mathfrak{S} := \{V(e) : e \in E(C)\}$ and F be an $\mathfrak{S}$-fragment. If $|F| \geq 3$, $|V(C) \cap F| = 2$ and $|E(C) \cap E_G(T_F, F)| = 2$, then*

1. *$E(C \cap G[T_F \cup F]) = \{x_1x_2, x_2x_3, x_3x_4, x_4x_5\}$, where $x_1, x_2, x_5 \in T_F$ and $x_3, x_4 \in F$.*
2. *For each vertex $a \in F \setminus \{x_3, x_4\}$, $a \notin C$.*
3. *For each vertex $a \in F \setminus \{x_3, x_4\}$, $N_G(a) = x_3 \cup (T_F \setminus x_2)$. In particular, a has degree $\kappa(G)$, $ax_1, ax_3, ax_5 \in E(G)$, and $ax_2, ax_4 \notin E(G)$.*
4. *$F - x_3 - x_4$ is independent.*
5. *$N_G(x_4) = x_3 \cup (T_F \setminus x_2)$.*
6. *x_1x_2 is the only edge of C that is contained in $G[T_F]$.*
7. *Every vertex in T_F lies in C.*

Proof. Let uv be an edge in $E(C \cap G[T_F])$ and $E(C) \cap E_G(T_F, F) = \{xx', yy'\}$, where $x, y \in T_F$ and $x', y' \in F$. If $x = y$, then $\{u, v\} \cap \{x, y\} = \emptyset$. By considering the edges xx', yy' and uv in C, we have $|E(C) \cap E_G(T_F, F)| > 2$. Hence, $x \neq y$. If $x' = y'$, then C contains a vertex in F other than x', which implies $|E(C) \cap E_G(T_F, F)| > 2$. Therefore, $x' \neq y'$ and $V(C) \cap F = \{x', y'\}$. Since $|V(C) \cap F| = 2$ and $|E(C) \cap E_G(T_F, F)| = 2$, $x'y' \in E(C)$. If $\{x, y\} = \{u, v\}$, then $C = xx'y'y$ and $V(C) \cap \overline{F} = \emptyset$ contradict Lemma 5. Hence, $\{x, y\} \neq \{u, v\}$.

Let a be any vertex in $F \setminus \{x', y'\}$. Note that $a \notin C$, since $|V(C) \cap F| = 2$. Consider any a-$(T_F \cup \{x', y'\})$ $\kappa(G)$-fan H. Note that $H \subseteq G[F \cup T_F]$, $V(C) \cap (F \cup T_F) \subseteq T_F \cup \{x', y'\}$ and $(H \setminus (T_F \cup \{x', y'\})) \cap C = \emptyset$. For any $z \in V(H) \cap (T_F \cup \{x', y'\})$, denote the path in H joining a to z by P_z. Suppose $x', y' \in H$. We can replace $x'y'$ in C by $x'P_{x'}aP_{y'}y'$ to construct a longer cycle. Suppose $x', y' \notin H$. Then $u, v \in H$. We can replace uv in C by uP_uaP_vv to construct a longer cycle. Hence, without loss of generality, suppose $x' \in H$ and $y' \notin H$. If $x \in H$, then we can replace xx' in C by $xP_xaP_{x'}x'$ to construct a longer cycle. So $x \notin H$ and $T_F \setminus \{x\} \subseteq H$. Since u, v cannot both belong to H, we have $x \in \{u, v\}$ and thus $y \notin \{u, v\}$. Now, assume $x = v$. Then the path $uxx'y'y$ lies in C. Since we can replace uxx' in C by $uP_u aP_{x'}x'$ and replace $x'y'y$ in C by $x'P_{x'}aP_y y$, the a-u, a-x', a-y paths in H are edges au, ax', ay respectively. This implies $ax, ay' \notin E(G)$. Suppose $u'v'$ is an edge in $E(C \cap G[T_F])$ other than uv. Then we can replace $u'v'$ by $u'P_{u'}aP_{v'}v'$ to construct a longer cycle. Hence, uv is the only edge in C that lies in $G[T_F]$.

Let a' be any vertex in $F \setminus \{x', y', a\}$ if exists. Consider any a'-$(T_F \cup \{x', y'\})$ $\kappa(G)$-fan H'. From the previous paragraph, either $x' \notin H$ and $y' \in H$, or $x' \in H$ and $y' \notin H$. Suppose $x' \notin H'$ and $y' \in H'$. Then $y \notin H'$ implying $u, v \in H'$, which is impossible. Hence, $x' \in H'$ and $y' \notin H'$. Arguing as above, the a'-u, a'-x', a'-y paths in H' are edges $a'u, a'x', a'y$ respectively, and $a'x, a'y \notin E(G)$. If aa' is an edge, then we can replace uxx' in C by $uaa'x'$ to construct a longer cycle. Hence, a and a' are not adjacent. Therefore, $F - x' - y'$ is independent, and $N_G(a) = x' \cup (T_F \setminus \{x\})$ for all vertices a in $F \setminus \{x', y'\}$. If xy' is an edge, then we can replace $uxx'y'y$ in C by $uxy'x'ay$ to construct a longer cycle. Hence, $N_G(y') = x' \cup (T_F \setminus \{x\})$. If there exists a vertex b in T_F that does not lie in C, we can replace $uxx'y'y$ in C by $uxx'y'bay$ to construct a longer cycle. Therefore, every vertex in T_F lies in C. Define $x_1 := u, x_2 := v = x, x_3 := x', x_4 := y', x_5 := y$ and the results (1)-(7) follow. $\square$

For any connected graph non-isomorphic to K_2, every edge is contractible. For any 2-connected graph non-isomorphic to K_3, it is obvious that every edge in a longest cycle is contractible. Results for contractible edges in a longest cycle of 3-connected graphs were given in the Introduction. For the rest of the paper, we will consider k-connected graphs ($k \geq 3$) that are triangle-free or have minimum degree at least $\frac{3}{2}k - 1$. As we will see, this leads us naturally to the following family of graphs.

For every $l \geq 4$ and every even $k \geq 4$, let $G_{l,k}$ be the lexicographic graph product $C_l \times K_{k/2}$. Clearly, $G_{l,k}$ is Hamiltonian, has connectivity k and is $(\frac{3}{2}k - 1)$-regular. Let C be a Hamiltonian cycle of G in which the vertices of every copy of $K_{k/2}$ are consecutive. Since all edges of every copy of $K_{k/2}$ are non-contractible, C contains exactly l contractible edges.

We will prove that with the exception of $G_{4,k}$ and $G_{5,k}$, the number of contractible edges in a longest cycle C is at least $\min\{|E(C)|, 6\}$.

Theorem 7. *Let G be a k-connected graph $(k \geq 3)$ that is triangle-free, or has minimum degree at least $\frac{3}{2}k - 1$ such that $G \ncong G_{4,k}$ and $G \ncong G_{5,k}$. For every longest cycle C of G, either all edges in C are contractible or C contains at least six contractible edges.*

Proof. If all edges in C are contractible, then we are done. Suppose C contains a non-contractible edge and define $\mathfrak{S} := \{V(e) : e \in E(C)\}$. Then G has connectivity k. Suppose G is triangle-free, or has minimum degree at least $\frac{3}{2}k - 1$ such that k is odd. Let B be any $\mathfrak{S}$-end. By Lemmas 2 and 3, $|B| > \frac{k}{2}$. In particular, $|B| \geq 2$ and thus $|V(C) \cap B| > 1$ by Lemma 5. This implies $|E(C) \cap (E_G(B, T_B) \cup E(B))| \geq 3$. By Lemma 4, all edges in $E(C) \cap (E_G(B, T_B) \cup E(B))$ are contractible. Consider an $\mathfrak{S}$-end B' in $\overline{B}$. Note that $(E_G(B', T_{B'}) \cup E(B')) \subseteq$

$(E_G(\overline{B}, T_B) \cup E(\overline{B}))$ and $(E_G(B, T_B) \cup E(B)) \cap (E_G(\overline{B}, T_B) \cup E(\overline{B})) = \emptyset$. Hence, C contains at least six contractible edges.

From now on, assume that G has minimum degree at least $\frac{3}{2}k - 1$ such that k is even ($k \geq 4$). Suppose C contains at most five contractible edges. We will divide the proof into a number of steps and show that $G \cong G_{4,k}$ or $G_{5,k}$.

(1) For every $\mathfrak{S}$-end B, $|V(C) \cap B| \geq 2$, $|E(C) \cap E(B)| \geq 1$, and $E(C) \cap E_G(B, T_B)$ contains exactly two edges, both of which are contractible.

Proof. By Lemma 3, B contains at least $\frac{k}{2} \geq 2$ vertices. By Lemma 5, $|V(C) \cap B| \geq 2$ and $|E(C) \cap E_G(B, T_B)| \geq 2$. Note that $|E(C) \cap E_G(B, T_B)|$ is even, and each edge e in $E(C) \cap E_G(B, T_B)$ is contractible by Lemma 4 as $|V(e) \cap B| = 1$. Since the same conclusion holds for any $\mathfrak{S}$-end in $\overline{B}$ and C contains at most five contractible edges, $|E(C) \cap E_G(B, T_B)| = 2$. Suppose $E(C) \cap E(B) = \emptyset$. Then, for any vertex in $V(C) \cap B$, its two neighbors in C must lie in T_B. Since $|V(C) \cap B| \geq 2$, this implies $|E(C) \cap E_G(B, T_B)| \geq 4$, which contradicts $|E(C) \cap E_G(B, T_B)| = 2$. Hence, $|E(C) \cap E(B)| \geq 1$. □

(2) If B is an $\mathfrak{S}$-end such that all edges in $E(C) \cap E(B)$ are contractible, then $|B| = 2$ and $k = 4$.

Proof. By (1), $|V(C) \cap B| \geq 2$ and the two edges in $E(C) \cap E_G(B, T_B)$ are contractible. If $|V(C) \cap B| \geq 3$, then $E(C) \cap E(B)$ has at least two edges, all of which are contractible by assumption. Combining with (1) on an $\mathfrak{S}$-end in $\overline{B}$, this implies that C has at least six contractible edges. Hence, $|V(C) \cap B| = 2$. If $|B| \geq 3$, then by Lemma 6, B contains a vertex of degree k, which is impossible as $\delta(G) \geq \frac{3}{2}k - 1$. Therefore, $|B| = 2$ and $k = 4$ by Lemma 3. □

(3) Suppose B is an $\mathfrak{S}$-end such that $E(C) \cap E(B)$ contains a non-contractible edge e. Let T be any k-separator containing $V(e)$. Then $B \subseteq T$, $T \cap T_B = \emptyset$, B is an $\mathfrak{S}$-atom, $G[B] \cong K_{\frac{k}{2}}$, and all edges in $E(B)$ are non-contractible. Also, every vertex in B is adjacent to every vertex in T_B, every vertex in B lies in C, and $C \cap G[B]$ is a Hamiltonian path of B.

Proof. By Lemma 4, $B \subseteq T$, $|B| = \frac{k}{2}$ and $T \cap T_B = \emptyset$. This implies all edges in $E(B)$ are non-contractible. By Lemma 3, B is an $\mathfrak{S}$-atom. Since $\delta(G) \geq \frac{3}{2}k - 1$, every vertex x in B is adjacent to every vertex in $(B \setminus \{x\}) \cup T_B$ and $G[B] \cong K_{\frac{k}{2}}$. Suppose there is a vertex x in B not contained in C. Let $e = uv$. We can replace uv in C by uxv to construct a longer cycle. Therefore, every vertex in B lies in C. Since $|E(C) \cap E_G(B, T_B)| = 2$, $C \cap G[B]$ is a Hamiltonian path of B. □

(4) Any two distinct $\mathfrak{S}$-ends are disjoint.

Proof. For $k \geq 6$, let B be any $\mathfrak{S}$-end. By (1), the two edges in $E(C) \cap E_G(B, T_B)$ are contractible and $|E(C) \cap E(B)| \geq 1$. By (2), not all edges in $E(C) \cap E(B)$ are contractible. By (3), $G[B] \cong K_{\frac{k}{2}}$, every vertex of B belongs to C, every edge in $E(C) \cap E(B)$ is non-contractible, and $C \cap G[B]$ is a Hamiltonian path of B. Consider two $\mathfrak{S}$-ends B_1 and B_2 such that $B_1 \cap B_2 \neq \emptyset$. For $i = 1, 2$, since every edge in $E(C)\cap E(B_i)$ is non-contractible and the two edges in $E(C)\cap E_G(B_i, T_{B_i})$ are contractible, the two Hamiltonian paths $C \cap G[B_1]$ and $C \cap G[B_2]$ must coincide implying $B_1 = B_2$.

For $k = 4$, by (2) and (3), every $\mathfrak{S}$-end is composed of either one contractible edge in C or one non-contractible edge in C. Let B and B' be two distinct $\mathfrak{S}$-ends. Denote the edge in $G[B]$ by e and the edge in $G[B']$ by e'. Suppose $B \cap B' \neq \emptyset$. Then $e \in E_G(B', T_{B'})$ and $e' \in E_G(B, T_B)$. By (1), e, e' are contractible. Hence, both B and B' are composed of one contractible edge with a common vertex. Denote $C \cap G[B \cup T_B] := x_1x_2x_3x_4$ where $B = \{x_2, x_3\}$, $C\cap G[B'\cup T_{B'}] := x'_1x'_2x'_3x'_4$ where $B' = \{x'_2, x'_3\}$, and $x'_2 = x_4, x'_3 = x_3, x'_4 = x_2$. Note that $x_3 = x'_3 = V(B) \cap V(B')$, $\{x_1, x_4\} \subseteq T_B$ and $\{x'_1, x'_4\} \subseteq T_{B'}$. Also, $x_1x_3, x_2x_4, x'_1x'_3, x'_2x'_4 \in E(G)$ as $\delta(G) \geq 5$. Since $x_4x'_1 = x'_1x'_2 \in E_G(B', T_{B'})$ is contractible by (1), $x'_1 \notin T_B$. Hence, $x'_1 \in \overline{B}$. But this is impossible as $x'_3 = x_3 \in B$ and $x'_1x'_3 \in E(G)$. Therefore, $B \cap B' = \emptyset$. $\square$

(5) If B and B' are two distinct $\mathfrak{S}$-ends not containing any contractible edge in C such that $E_G(B, B') \neq \emptyset$, then $B \subseteq T_{B'}$ and $B' \subseteq T_B$.

Proof. By (3), both $G[B]$ and $G[B']$ are $K_{\frac{k}{2}}$. By (4), B and B' are disjoint. Since $E_G(B, B') \neq \emptyset$, $B \cap T_{B'} \neq \emptyset$ and $B' \cap T_B \neq \emptyset$. Suppose $B \nsubseteq T_{B'}$. Then $B \cap \overline{B'} \neq \emptyset$. This implies $|B \cap T_{B'}| < |B| = \frac{k}{2}$ and $|B \cap \overline{B'}| < |B| = \frac{k}{2}$. By Lemma 3, $B \cap \overline{B'}$ is not a fragment and thus $\frac{k}{2} > |B \cap T_{B'}| > |B' \cap T_B|$. This implies $B' \cap \overline{B} = \emptyset$ and $B' = B' \cap T_B$. But then $\frac{k}{2} > |B \cap T_{B'}| = |B'| = \frac{k}{2}$, which is impossible. Therefore, $B \subseteq T_{B'}$ and by symmetry, $B' \subseteq T_B$. $\square$

(6) Every $\mathfrak{S}$-end does not contain any contractible edge in C.

Proof. Suppose B is an $\mathfrak{S}$-end containing a contractible edge in C. By (3), all edges in $E(C) \cap E(B)$ are contractible. By (2), $k = 4$ and $|B| = 2$. By (1), $|E(C) \cap E(B)| = 1$. Let A be an $\mathfrak{S}$-end in $\overline{B}$. Then $A = A \cap \overline{B}$ and $A \cap T_B = \emptyset$. By Lemma 1, $|A \cap T_B| \geq |B \cap T_A|$ implying $B \cap T_A = \emptyset$ and $B \subseteq \overline{A}$. Note that $(E(B) \cup E_G(B, T_B)) \cap E_G(A, T_A) = \emptyset$. By (1), the five edges in $E(C) \cap (E_G(A, T_A) \cup E_G(B, T_B) \cup E(B))$ are contractible. Since C has at most five contractible edges, all edges in $E(C) \cap E(A)$ are non-contractible. By (3), A is an $\mathfrak{S}$-atom K_2 contained in a 4-separator T such that $T \cap T_A = \emptyset$. This implies $A \subseteq \overline{B} \cap T$.

Since the edge in $E(C)\cap E(B)$ is contractible, $B \nsubseteq T$. Let F be a T-fragment such that $B \cap F \neq \emptyset$ and B' be an $\mathfrak{S}$-end in $\overline{F}$. Suppose $B \subseteq F$. We have $(E(B) \cup E_G(B, T_B)) \cap E_G(A, T_A) = \emptyset$, $(E(B) \cup E_G(B, T_B)) \cap E_G(B', T_{B'}) = \emptyset$ and $E(C) \cap E_G(B', T_{B'}) \neq E(C) \cap E_G(A, T_A)$. But then $E(C) \cap (E_G(B, T_B) \cup E(B)\cup E_G(B', T_{B'})\cup E_G(A, T_A))$ contains at least six contractible edges, which is impossible. Hence, $B\cap F \neq \emptyset$ and $B\cap T \neq \emptyset$. Since $|B| = 2$, $|B\cap F| = 1 = |B\cap T|$.

Let D be an $\mathfrak{S}$-end in F and D' be an $\mathfrak{S}$-end in $\overline{F}$. By (4), A, D, B, D' are pairwise disjoint. Since $(E(B) \cup E_G(B, T_B)) \cap E_G(A, T_A) = \emptyset$ and C contains at most five contractible edges, $G[D \cup D']$ does not contain any contractible edges in C. By (1), (3) and $k = 4$, D, D' are both $\mathfrak{S}$-atoms K_2, and $E(D), E(D')$ both consist of a non-contractible edge in C. Since C contains at most five contractible edges, each of $\{A, D\}$, $\{D, B\}$, $\{B, D'\}$ and $\{D', A\}$ is connected by exactly one contractible edge in C. Note that $V(C) = V(A \cup D \cup B \cup D')$. By (5), $D \subseteq T_A$ and $D' \subseteq T_A$. Since $k = 4$, $T_A = V(D \cup D')$ and $T \cap T_A = \emptyset$. Recall that $A \subseteq \overline{B} \cap T$ and $|B \cap T| = 1$. Let x be the vertex in $T \setminus (A \cup B)$ which lies in $T \cap \overline{A}$. As $V(C) = V(A \cup D \cup B \cup D')$, $x \notin C$. Define $P := C \cap G[B \cup D \cup D']$ which is a path of six vertices in $G[\overline{A} \cup T_A]$. Let H be an x-P 4-fan in $G[\overline{A} \cup T_A]$. Then there is an edge uv in P such that $u, v \in H$. Let P_u be the path in H joining x and u, and P_v be the path in H joining x and v. Now, we can replace uv by uP_uxP_vv to construct a longer cycle than C, which is impossible. □

(7) $G \cong G_{4,k}$ or $G_{5,k}$.

Proof. Consider any $\mathfrak{S}$-end B. By (6), all edges in $E(C) \cap E(B)$ are non-contractible. By (3), B is contained in a k-separator T such that $G[B] \cong K_{\frac{k}{2}}$ and $T \cap T_B = \emptyset$. Let F be a T-fragment. Since C contains at most five contractible edges, there exists an $\mathfrak{S}$-end in F or $\overline{F}$, say B', such that $E_G(B, T_B) \cap E_G(B', T_{B'}) \cap E(C) \neq \emptyset$. By (5), $B \subseteq T_{B'}$ and $B' \subseteq T_B$. By $\delta(G) \geq \frac{3k}{2} - 1$, (6) and (3), $G[B'] \cong K_{\frac{k}{2}}$ and $G[B \cup B'] \cong K_k$.

Let D be an $\mathfrak{S}$-end in $\overline{B}$ and D' be an $\mathfrak{S}$-end in $\overline{B'}$. Suppose $D \neq D'$. By (4), B, B', D, D' are disjoint. Since C contains at most five contractible edges, there exists a set of four pairwise disjoint $\mathfrak{S}$-ends A_1, A_2, A_3, A_4 such that $\{A_1, A_2, A_3, A_4\} = \{B, B', D, D'\}$ and A_i is connected to A_{i+1} by a contractible edge in C for $i = 1, 2, 3$. By (6) and (3), all $G[A_i]$'s are $K_{\frac{k}{2}}$, all edges in $E(A_i)$ are non-contractible, and $C \cap G[A_i]$ is a Hamiltonian path of A_i. By (5), $A_1 \subseteq T_{A_2}$, $A_3 \subseteq T_{A_2}$, $A_2 \subseteq T_{A_3}$ and $A_4 \subseteq T_{A_3}$. Therefore, $T_{A_2} = A_1 \cup A_3$ and $T_{A_3} = A_2 \cup A_4$. Suppose $D = D'$. Then $D \subseteq \overline{B} \cap \overline{B'}$ and $B \cup B' \subseteq \overline{D}$. By applying (6) and (3) to D, let S be a k-separator containing D such that $S \cap T_D = \emptyset$. Recall that $G[B \cup B'] \cong K_k$. Let X be an S-fragment such that $B \cup B' \subseteq \overline{D} \cap (X \cup S)$. Then $\overline{X}$ contains an $\mathfrak{S}$-end D' different from D, B, B'. Again, we have the same conclusion as the case $D \neq D'$.

If $\overline{A_2} \cap \overline{A_3} = \emptyset$, then $G \cong G_{4,k}$. If $\overline{A_2} \cap \overline{A_3} \neq \emptyset$, then $\overline{A_2} \cap \overline{A_3}$ is an $\mathfrak{S}$-fragment and contains an $\mathfrak{S}$-end, say A. By (6) and (3), $G[A]$ is $K_{\frac{k}{2}}$, all edges in $E(A)$ are non-contractible, and $C \cap G[A]$ is a Hamiltonian path of A. Since C contains at most five contractible edges, A is connected to A_1 by a contractible edge in C and A is connected to A_4 by a contractible edge in C. Denote $Z := A_1 \cup A_2 \cup A_3 \cup A_4 \cup A$. Then $V(C) = Z$. Suppose x is a vertex in $G \setminus Z \subseteq \overline{A_2} \cap \overline{A_3}$. Consider an x-$(A_1 \cup A_4 \cup A)$ k-fan H. Since $k \geq 4$, two of the x-$(A_1 \cup A_4 \cup A)$ paths in H end in the same $\mathfrak{S}$-end and we can use them to construct a longer cycle. Therefore, $G \setminus Z = \emptyset$ and $G \cong G_{5,k}$. □

This completes the proof of the theorem. □

4 Upper Bounds for $f(k)$

In the previous section, we proved $f(k) \geq 6$ with the only exception that the graph is $G_{4,k}$ or $G_{5,k}$. To find an upper bound for $f(k)$, we just need to examine a particular k-connected graph and see if there is a longest cycle containing a non-contractible edge. If such longest cycle C exists and the number of contractible edges in C is l, then $f(k) \leq l \leq |C| - 1$. Here, we exhibit an infinite family of triangle-free 3-connected graphs in which there is a longest cycle containing exactly seven contractible edges (in fact, $f(3) = 7$ for triangle-free graphs as demonstrated in the next section), generalize this family to every odd $k \geq 3$ showing that $2k + 1$ is an upper bound for $f(k)$, and then generalize this family to arbitrary k.

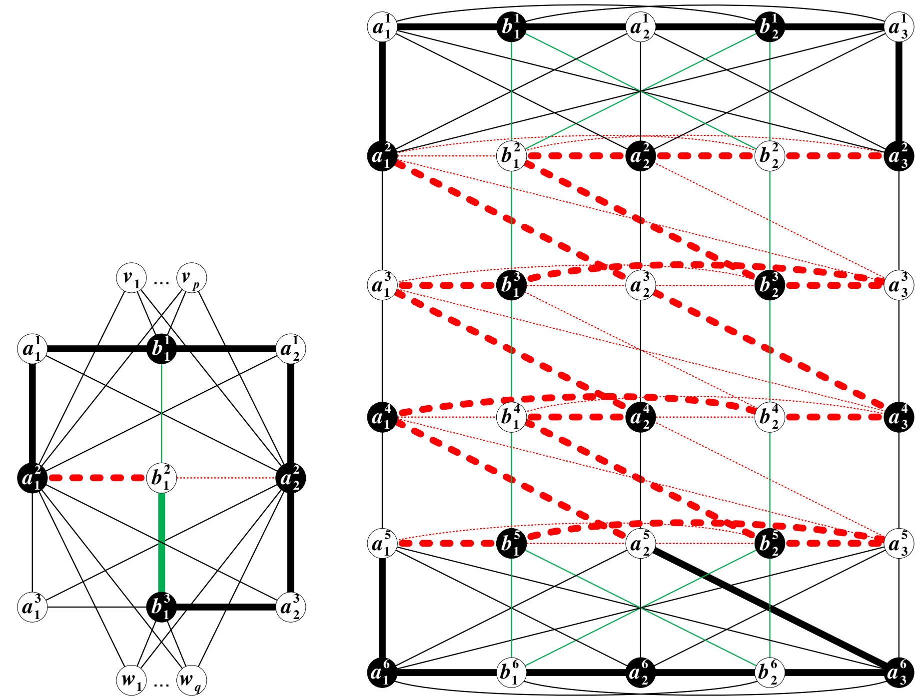
We will use a construction that is similar to the one given in [10]. For $k \geq 2$, $l \geq 3$ and $p, q \geq 0$, let $A_1, \ldots, A_l, B_1, \ldots, B_l$ be pairwise disjoint sets such that $A_1 := \{a^1_1, a^1_2, \ldots, a^1_{\lceil k/2 \rceil}, v_1, v_2, \ldots, v_p\}$ where $a^1_{\lceil k/2 \rceil + i} = v_i$ for $1 \leq i \leq p$, $A_l := \{a^l_1, a^l_2, \ldots, a^l_{\lceil k/2 \rceil}, w_1, w_2, \ldots, w_q\}$ where $a^l_{\lceil k/2 \rceil + i} = w_i$ for $1 \leq i \leq q$, $A_h := \{a^h_1, a^h_2, \ldots, a^h_{\lceil k/2 \rceil}\}$ for every $1 < h < l$, and $B_h := \{b^h_1, b^h_2, \ldots, b^h_{\lfloor k/2 \rfloor}\}$ for every $1 \leq h \leq l$. Let $G_{k,l,p,q}$ be the graph (see Figure 1) with vertex set $\bigcup_{h=1}^{l} (A_h \cup B_h)$ and edge set

$$\begin{aligned} E(G_{k,l,p,q}) := &\{a^h_i b^h_j : 1 \leq h \leq l, 1 \leq i \leq |A_h|, 1 \leq j \leq |B_h|\} \cup \\ &\{a^1_i a^2_j : 1 \leq i \leq |A_1|, 1 \leq j \leq |A_2|\} \cup \\ &\{b^1_i b^2_j : 1 \leq i \leq |B_1|, 1 \leq j \leq |B_2|\} \cup \\ &\{a^{l-1}_i a^l_j : 1 \leq i \leq |A_{l-1}|, 1 \leq j \leq |A_l|\} \cup \\ &\{b^{l-1}_i b^l_j : 1 \leq i \leq |B_{l-1}|, 1 \leq j \leq |B_l|\} \cup \\ &\{a^h_i a^{h+1}_j : 2 \leq h \leq l-2, 1 \leq i \leq j \leq |A_h|\} \cup \\ &\{b^h_i b^{h+1}_j : 2 \leq h \leq l-2, 1 \leq i \leq j \leq |B_h|\}. \end{aligned}$$

Thus, the vertices of $G_{k,l,p,q}$ are partitioned into l *levels*, of which the first two contain the complete bipartite subgraphs $K_{|A_1|,|A_2|}$ and $K_{|B_1|,|B_2|}$ (induced by the vertex sets $A_1 \cup A_2$ and $B_1 \cup B_2$, respectively), and the last two contain the complete bipartite subgraphs $K_{|A_{l-1}|,|A_l|}$ and $K_{|B_{l-1}|,|B_l|}$. Note that the remaining pairs of consecutive levels induce proper subgraphs of these complete bipartite graphs. By construction, $G_{k,l,p,q}$ is bipartite, and it is not hard to verify that $G_{k,l,p,q}$ is k-connected, and that the non-contractible edges of $G_{k,l,p,q}$ are exactly the edges in

$$\begin{aligned} &\{a^h_i b^h_j : 2 \leq h \leq l-1, 1 \leq i \leq |A_h|, 1 \leq j \leq |B_h|\} \cup \\ &\{a^h_i a^{h+1}_j : 2 \leq h \leq l-2, 1 \leq i < j \leq |A_h|\} \cup \\ &\{b^h_i b^{h+1}_j : 2 \leq h \leq l-2, 1 \leq i < j \leq |B_h|\}. \end{aligned}$$

Lemma 8. *Let $k \geq 3$ be odd and $p, q \geq 0$. Then $G_{k,3,p,q}$ is k-connected, bipartite and non-Hamiltonian, and has a longest cycle that contains exactly $2k + 1$ contractible edges.*



(a) The family of graphs $G_{3,3,p,q}$ on $9+p+q$ vertices, in which every longest cycle has length 8. Red dotted edges depict the non-contractible edges of $G_{3,3,p,q}$, and fat edges depict a longest cycle that contains exactly $2k+1 = 7$ contractible edges.

(b) The Hamiltonian graph $G_{5,6,0,0}$ on 30 vertices; fat edges depict a Hamiltonian cycle that contains exactly $2k+2 = 12$ contractible edges. There are non-Hamiltonian 5-connected graphs and longest cycles of these graphs that contain exactly $2k+1 = 11$ contractible edges, for example the graph $G_{5,3,0,0}$.

Figure 1: Two examples of the graphs $G_{k,l,p,q}$ for $k = 3$ and $k = 5$.

Proof. The graph $G_{k,3,p,q}$ is bipartite and has $3k + p + q$ vertices, so that the smaller color class, say black, consists of exactly $k + \lfloor k/2 \rfloor = (3k-1)/2$ vertices. Hence, any longest cycle of $G_{k,3,p,q}$ has length at most $3k-1$. The cycle $a_1^1, b_1^1, \ldots, a_{|A_2|}^1, a_{|A_2|}^2, a_{|A_2|}^3, \ldots, b_1^3, b_{|B_2|}^2, \ldots, a_1^2, a_1^1$ (see Figure 1a) of length $3k-1$ is therefore a longest cycle, and contains exactly $3k-1-(k-2) = 2k+1$ contractible edges. □

Since $p, q \geq 0$ are arbitrary, Lemma 8 gives an infinite graph family that attains the upper bound $2k+1$ for every odd $k \geq 3$ and is triangle-free. While we will show in section 6 that restricting the cycle spectrum of 3-connected graphs further to girth 5 gives an increase of the constant lower bound on the number of contractible edges to a linear function depending on $|E(C)|$, this cannot be expected from avoiding all odd cycles, as all graphs for our upper bounds are bipartite (the same holds for $k > 3$).

For even $k \geq 3$, we now provide infinite families of k-connected triangle-free graphs that prove the slightly weaker upper bound $2k + 2$.

Lemma 9. *Let $k \geq 4$ and $l \geq 3$ such that l is even if k is odd. Then $G_{k,l,0,0}$ is k-connected, bipartite and Hamiltonian, and has a longest cycle that contains exactly $2k + 2$ contractible edges.*

Proof. The graph $G_{k,l,0,0}$ is bipartite and has lk vertices. If k is even, the cycle C consisting of the path $a_1^2, a_1^1, b_1^1, \ldots, b_{|B_1|}^1, b_{|B_2|}^2, \ldots, b_1^2$, the path $b_2^{l-1}, \ldots, b_{|B_{l-1}|}^{l-1}, b_{|B_l|}^l, \ldots, a_1^l a_1^{l-1} b_1^{l-1} a_2^{l-1}$ on the last two levels, and the two paths $a_1^h, a_2^{h+1}, b_1^{h+1}$ and $b_1^h, b_2^{h+1}, \ldots, b_{|B_{h+1}|}^{h+1}, a_1^{h+1}$ for every two consecutive levels $1 < h < h + 1 < l$ is Hamiltonian. Since the contractible edges of C are exactly the ones that intersect the first or the last level, C contains exactly $2k+2$ contractible edges.

If k is odd, it is not possible to construct a longest cycle such that every two consecutive mid-levels induce the same pattern. However, since l is even in this case (otherwise, $G_{k,l,0,0}$ would not be Hamiltonian), we may use two patterns for the mid-levels. Let C be the cycle (see Figure 1b) that consists of the path $a_1^2, a_1^1, b_1^1, \ldots, a_{|A_1|}^1, a_{|A_2|}^2, \ldots, b_1^2$, the path $a_2^{l-1}, a_{|A_l|}^l, \ldots, a_1^l, a_1^{l-1}$, the paths a_1^h, a_2^{h+1} and $b_1^h, b_2^{h+1}, \ldots, a_{|A_{h+1}|}^{h+1}, b_1^{h+1}, a_1^{h+1}$ for every even $1 < h < l$, and the paths $a_1^h, a_2^{h+1}, b_1^{h+1}$ and $a_2^h, a_{|A_{h+1}|}^{h+1}, \ldots, b_2^{h+1}, a_1^{h+1}$ for every odd $1 < h < l$. Clearly, C is Hamiltonian and contains exactly $2k + 2$ contractible edges. □

For Hamiltonian triangle-free k-connected graphs (k is odd), suppose a longest cycle C contains a non-contractible edge. Then the intersection of C with any $\mathfrak{S}$-end ($\mathfrak{S} := \{V(e) : e \in E(C)\}$) has at least $k + 1$ edges by Lemma 2, each of which is contractible by Lemma 4. This gives a lower bound of $2k + 2$ for $f(k)$ that matches the upper bound of Lemma 9, and hence the following theorem.

Theorem 10. *Let G be a Hamiltonian triangle-free k-connected graph such that k is odd. Then every longest cycle C of G contains at least* $\min\{|E(C)|, 2k + 2\}$ *contractible edges, and the graphs $G_{k,l,0,0}$ show that this bound is best possible.*

The following lemma gives upper bounds for $f(k)$ for graphs with minimum degree $\lfloor 3k/2 \rfloor - 1$. Let $H_{k,l,p,q}$ be the graph obtained from $G_{k,l,p,q}$ by adding an edge between every two non-adjacent vertices in $A_h \cup B_h$ for every level $1 \leq h \leq l$. Clearly, $H_{k,l,p,q}$ is not bipartite.

Lemma 11. *Let $k \geq 4$ and $l \geq 3$ such that l is even if k is odd. Then $H_{k,l,0,0}$ is k-connected, Hamiltonian and has minimum degree $\lfloor 3k/2 \rfloor - 1$ and a longest cycle that contains exactly $2k + 2$ contractible edges.*

Proof. We show that $H_{k,l,0,0}$ has minimum degree $\lfloor 3k/2 \rfloor - 1$. The other claims follow directly from noting that the contractability and non-contractability of the edges in $G_{k,l,0,0}$ is preserved in $H_{k,l,0,0}$, that the only new contractible edges are the ones intersecting the first or the last level, and from taking exactly the same cycles as in the proof of Lemma 9.

Consider $H_{k,l,0,0}$. Every vertex in $A_1 \cup B_1 \cup A_l \cup B_l$ has degree at least $k - 1 + \lfloor k/2 \rfloor = \lfloor 3k/2 \rfloor - 1$, and the vertices of $B_1 \cup B_l$ attain this degree. By symmetry and $l \geq 3$, every vertex of $A_2 \cup B_2 \cup A_{l-1} \cup B_{l-1}$ has degree at least $\lfloor 3k/2 \rfloor$. For every $3 \leq h \leq l$ and every $1 \leq i \leq |A_h|$, a_i^h has at least i neighbors in A_{h-1}, exactly $k-1$ neighbors in $A_h \cup B_h$, and at least $\lceil k/2 \rceil - i + 1$ neighbors in A_{h+1}. Hence, a_i^h has degree at least $\lceil 3k/2 \rceil$. Similarly, for every $3 \leq h \leq l$ and every $1 \leq i \leq |B_h|$, b_i^h has at least i neighbors in B_{h-1}, exactly $k-1$ neighbors in $A_h \cup B_h$, and at least $\lfloor k/2 \rfloor - i + 1$ neighbors in B_{h+1}. Hence, b_i^h has degree at least $\lfloor 3k/2 \rfloor$, which gives the claim. $\square$

5 Triangle-free 3-Connected Graphs

This section investigates the number of contractible edges in a longest cycle C of a triangle-free 3-connected graph. Theorem 7 tells us that the lower bound is $\min\{|E(C)|, 6\}$. With a little extra effort, we can improve the bound to $\min\{|E(C)|, 7\}$. Also, we will characterize all triangle-free 3-connected graphs having a longest cycle containing exactly six/seven contractible edges.

Theorem 12. *Let G be a triangle-free 3-connected graph and C be a longest cycle in G. If C contains a non-contractible edge, then C has at least seven contractible edges. If C contains more than one non-contractible edges, then C has at least eight contractible edges.*

Proof. Suppose C contains a non-contractible edge. Define $\mathfrak{S} := \{V(e) : e \in E(C)\}$ and let B be an $\mathfrak{S}$-end. Define $R_B := E(C) \cap (E_G(B, T_B) \cup E(B))$. By Lemmas 2 and 5, R_B contains at least three edges, all of which are contractible by Lemma 4. Since $\overline{B}$ contains an $\mathfrak{S}$-end, C has at least six contractible edges. Suppose for every $\mathfrak{S}$-end B, $|R_B| \geq 4$. Then C has at least eight contractible edges. Therefore, assume that there is an $\mathfrak{S}$-end B such that $|R_B| = 3$. By Lemma 6, $B \cup T_B$ has the following properties.

1. $E(C \cap G[T_B \cup B]) := \{x_1x_2, x_2x_3, x_3x_4, x_4x_5\}$, where $x_1, x_2, x_5 \in T_B$ and $x_3, x_4 \in B$. In fact, $T_B = \{x_1, x_2, x_5\}$.
2. For each vertex $a \in B \setminus \{x_3, x_4\}$, $a \notin C$.
3. For each vertex $a \in B \setminus \{x_3, x_4\}$, $N_G(a) = x_3 \cup (T_B \setminus x_2) = \{x_1, x_3, x_5\}$. In particular, a has degree 3, $ax_1, ax_3, ax_5 \in E(G)$, and $ax_2, ax_4 \notin E(G)$.
4. $B - x_3 - x_4$ is independent.
5. $N_G(x_4) = x_3 \cup (T_B \setminus x_2) = \{x_1, x_3, x_5\}$.
6. x_1x_2 is the only edge of C that is contained in $G[T_B]$.
7. Every vertex in T_B lies in C.

Let x_0 be the neighbor of x_1 in C other than x_2, and x_6 be the neighbor of x_5 in C other than x_4. By Lemma 5, $x_0 \neq x_6$. For later use, denote x_{-1} to be the neighbor of x_0 in C other than x_1.

Claim 1. Suppose x_0x_1 is non-contractible and let S be any 3-separator containing x_0x_1. Then x_2, x_3, x_4, x_5, x_6 lie in the same S-fragment D such that $B \subseteq D$, $\overline{D} \subseteq \overline{B}$, $x_3, x_4 \in D \cap B$, $x_2, x_5 \in D \cap T_B$ and $x_6 \in D \cap \overline{B}$.

Proof. Recall that $T_B = \{x_1, x_2, x_5\}$, $x_3, x_4 \in B$, $x_0, x_6 \in \overline{B}$, and a is any vertex in $B \setminus \{x_3, x_4\}$. Suppose $x_2 \in S$. Let D be an S-fragment containing x_5. Then $B \cap S = \overline{D} \cap T_B = \emptyset$. This implies $B \subseteq D$ and $\overline{D} \subseteq \overline{B}$. But then $C \cap \overline{D} = \emptyset$, which contradicts Lemma 5. Therefore, $x_2 \notin S$.

Let D be an S-fragment containing x_2. Suppose $x_5 \in S$. Then $B \cap S = \overline{D} \cap T_B = \emptyset$. This implies $B \subseteq D$ and $\overline{D} \subseteq \overline{B}$. We must have $x_6 \in \overline{D} \cap \overline{B}$ and $C \cap D \cap \overline{B} = \emptyset$. Since x_0x_2 is not an edge, $D \cap \overline{B} \neq \emptyset$. Let u be a vertex in $D \cap \overline{B}$. Consider a u-$\{x_0, x_1, x_2, x_5\}$ 3-fan H. Then $x_1 \notin H$ for otherwise we can use H to construct a longer cycle. Denote the u-x_0 path and the u-x_2 path in H by P_0 and P_2 respectively. Now, $x_0P_0uP_2x_2x_3ax_1x_4x_5x_6Cx_0$ is a longer cycle, which is a contradiction. Suppose $x_5 \in \overline{D}$. If $x_3 \in B \cap D$, then $x_4 \in B \cap S$. But this is impossible, since a is adjacent to x_1, x_3, x_5. So, $x_3 \in B \cap S$ and $x_4 \in B \cap \overline{D}$. Now, $B \cap D = \emptyset$ and $C \cap D \cap \overline{B} = \emptyset$. Since x_0x_2 is not an edge, $D \cap \overline{B} \neq \emptyset$. Let u be a vertex in $D \cap \overline{B}$. Then we can use a u-$\{x_0, x_1, x_2\}$ 3-fan to construct a longer cycle. Therefore, $x_5 \in D$. If $B \cap S \neq \emptyset$, then $\overline{D} = \emptyset$, which is impossible. Hence, $B \cap S = \emptyset$. This implies $B \subseteq D$ and $\overline{D} \subseteq \overline{B}$. We have $x_3, x_4 \in D \cap B$.

Let y be the vertex in $S \setminus \{x_1, x_0\}$. Then $y \in \overline{B} \cap S$. We have $C \cap \overline{D} \neq \emptyset$, $y \in C$ and $x_6 \in (D \cap \overline{B}) \cup \{y\}$. Suppose $x_6 = y$. If $D \cap \overline{B} = \emptyset$, then since G is triangle-free and 3-connected, $x_2x_6 \in E(G)$. But, $x_0x_1ax_5x_4x_3x_2x_6Cx_0$ is a longer cycle. Let $u \in D \cap \overline{B}$. Consider a u-$\{x_0, x_1, x_2, x_5, x_6\}$ 3-fan H. Note that exactly one vertex of $\{x_0, x_1\}$, one of $\{x_1, x_2\}$ and one of $\{x_5, x_6\}$ is contained in H. This implies that $x_0 \in H$ and $x_2 \in H$. Denote the u-x_0 path in H by P_0 and the u-x_2 path in H by P_2. But, $x_0P_0uP_2x_2x_3ax_1x_4x_5x_6Cx_0$ is a longer cycle. Therefore, $x_6 \in D \cap \overline{B}$. □

Claim 2. Suppose x_5x_6 is non-contractible and let T be any 3-separator containing x_5x_6. Then x_0, x_1, x_2, x_3, x_4 lie in the same T-fragment F such that $B \subseteq F$, $\overline{F} \subseteq \overline{B}$, $x_3, x_4 \in F \cap B$, $x_1, x_2 \in F \cap T_B$ and $x_0 \in F \cap \overline{B}$.

Proof. Recall that $T_B = \{x_1, x_2, x_5\}$, $x_3, x_4 \in B$, $x_0, x_6 \in \overline{B}$, and a is any vertex in $B \setminus \{x_3, x_4\}$. Let F be a T-fragment intersecting x_1x_2. Suppose $|F \cap \{x_1, x_2\}| = 1$. Then $|T \cap \{x_1, x_2\}| = 1$. We have $x_0 \notin T$ and $B \cap T = \overline{F} \cap T_B = \emptyset$. Hence, $B \subseteq F$ and $\overline{F} \subseteq \overline{B}$. If $x_0 \in F \cap \overline{B}$, then $C \cap \overline{F} = \emptyset$ which contradicts Lemma 5. Therefore, $x_0 \in \overline{F}$. This implies $x_1 \in T \cap T_B$ and $x_2 \in F \cap T_B$. Note that $C \cap (F \cap \overline{B}) = \emptyset$. If $F \cap \overline{B}$ contains a vertex u, then we can use a u-$\{x_1, x_2, x_5, x_6\}$ 3-fan to construct a longer cycle. Hence, $F \cap \overline{B} = \emptyset$. Since G is 3-connected, $x_2x_6 \in E(G)$. But $x_0x_1ax_5x_4x_3x_2x_6Cx_0$ is a longer cycle.

Therefore, $x_1, x_2 \in F$. If $B \cap T \neq \emptyset$, then $\overline{F} = \emptyset$, which is impossible. Hence, $B \cap T = \emptyset$. This implies $B \subseteq F$ and $\overline{F} \subseteq \overline{B}$. We have $x_3, x_4 \in F \cap B$. Let x be the vertex in $\overline{B} \cap T$ other than x_6. By Lemma 5, $C \cap \overline{F} \neq \emptyset$. We have $x \in C$ and $x_0 \in (F \cap \overline{B}) \cup \{x\}$. Suppose $x_0 = x$. If $F \cap \overline{B} = \emptyset$, then $x_2x_6 \in E(G)$ since G is triangle-free and 3-connected. But $x_0x_1ax_5x_4x_3x_2x_6Cx_0$ is a longer cycle. Let $u \in F \cap \overline{B}$ and consider a u-$\{x_0, x_1, x_2, x_5, x_6\}$ 3-fan H. Then exactly one vertex of $\{x_0, x_1\}$, one of $\{x_1, x_2\}$ and one of $\{x_5, x_6\}$ is contained in H. This implies that $x_0 \in H$ and $x_2 \in H$. Denote the u-x_0 path in H by P_0 and the u-x_2 path in H by P_2. Then $x_0P_0uP_2x_2x_3ax_1x_4x_5x_6Cx_0$ is a longer cycle. Therefore, $x_0 \in F \cap \overline{B}$. □

Now, we will consider the following four cases.

(I) Both x_0x_1 and x_5x_6 are non-contractible.

Let $S := \{y, x_0, x_1\}$ be a 3-separator containing x_0x_1 and $T := \{x, x_5, x_6\}$ be a 3-separator containing x_5x_6. By Claims 1 and 2, let D be an S-fragment containing $\{x_2, x_3, x_4, x_5, x_6\}$ and F be a T-fragment containing $\{x_0, x_1, x_2, x_3, x_4\}$.

(a) $y \in F \cap S$ and $x \in D \cap T$.

We have $\overline{D} \subseteq F$ and $\overline{F} \subseteq D$. By considering an $\mathfrak{S}$-end in $\overline{D}$ and an $\mathfrak{S}$-end in $\overline{F}$, C has at least nine contractible edges.

(b) $y = x \in S \cap T$.

We have $\overline{D} \subseteq F$ and $\overline{F} \subseteq D$. By considering an $\mathfrak{S}$-end in $\overline{D}$ and an $\mathfrak{S}$-end in $\overline{F}$, C has at least nine contractible edges.

(c) $y \in \overline{F} \cap S$ and $x \in \overline{D} \cap T$.

We have $S \cap T = \emptyset$ which implies $\overline{D} \cap \overline{F} = \emptyset$. By considering an $\mathfrak{S}$-end in $\overline{D}$ and an $\mathfrak{S}$-end in $\overline{F}$, C has at least eight contractible edges.

(II) x_0x_1 is non-contractible and x_5x_6 is contractible.

Let $S := \{x_0, x_1, y\}$ be a 3-separator containing x_0x_1. By Claim 1, let D be a S-fragment containing x_2, x_3, x_4, x_5, x_6 such that $x_3, x_4 \in D \cap B$, $x_2, x_5 \in D \cap T_B$ and $x_6 \in D \cap \overline{B}$. Note that $B \cap S = \overline{D} \cap T_B = B \cap \overline{D} = \emptyset$ and $y \in \overline{B} \cap S$. Consider an $\mathfrak{S}$-end B' in $\overline{D}$. If $|R_{B'}| \geq 4$, then C has at least eight contractible edges. Hence, assume $|R_{B'}| = 3$. By Lemma 6, denote $R_{B'} := \{x'_2x'_3, x'_3x'_4, x'_4x'_5\}$ where $x'_1, x'_2, x'_5 \in T_{B'}$, $x'_3, x'_4 \in B'$ and $x'_1x'_2 \in E(C)$. Let a' be a vertex in $B' \setminus \{x'_3, x'_4\}$. We will use the same notation for B' below whenever applicable. If there exists a contractible edge in $C \setminus (R_B \cup x_5x_6 \cup R_{B'})$, then C has at least eight contractible edges. Otherwise, by considering B', we have the case (I) unless $x'_1 = x_6$, $x'_2 = y$ and $x'_3, x'_4 \in \overline{D}$. Since $N_G(a') = \{x'_1, x'_3, x'_5\}$, a' must be x_0 or x_1 contradicting (2) of Lemma 6.

(III) x_0x_1 is contractible and x_5x_6 is non-contractible.

Let $T := \{x_5, x_6, x\}$ be a 3-separator containing x_5x_6. By Claim 2, let F be a T-fragment containing x_0, x_1, x_2, x_3, x_4 such that $x_3, x_4 \in F \cap B$, $x_1, x_2 \in F \cap T_B$

and $x_0 \in F \cap \overline{B}$. Note that $B \cap T = \overline{F} \cap T_B = B \cap \overline{F} = \emptyset$ and $x \in \overline{B} \cap T$. Consider an $\mathfrak{S}$-end B' in $\overline{F}$. If $|R_{B'}| \geq 4$, then C has at least eight contractible edges. Hence, assume $|R_{B'}| = 3$. If there exists a contractible edge in $C \setminus (R_B \cup x_0x_1 \cup R_{B'})$, then C has at least eight contractible edges (in particular, this includes the case when x_0x_{-1} is contractible). Otherwise, by considering B', we have the case (I) unless (a) $x'_1 = x_0$, $x'_2 = x$ and $x'_3, x'_4 \in \overline{F}$, or (b) $x'_1 = x_5$, $x'_2 = x_6$ and $x'_3, x'_4 \in \overline{F}$. For (a), since $N_G(a') = \{x'_1, x'_3, x'_5\}$, a' must be x_5 or x_6 contradicting (2) of Lemma 6. For (b), C has at least eight contractible edges since $x'_0x'_{-1} = x_4x_3$ is contractible.

(IV) Both x_0x_1 and x_5x_6 are contractible.

Let B' be an $\mathfrak{S}$-end in $\overline{B}$. Suppose C has exactly six contractible edges. Then $|R_{B'}| = 3$, and both x_0x_1 and x_5x_6 belong to $R_{B'}$. Therefore, $x'_1 = x_2, x'_2 = x_1, x'_3 = x_0, x'_4 = x_6, x'_5 = x_5$ and $C = x_0x_1x_2x_3x_4x_5x_6x_0$. But then $x_1ax_3x_4x_5a'x_0x_6x_2x_1$ is a longer cycle. We can conclude that C has at least seven contractible edges.

Suppose C has more than one non-contractible edges. If the cases (I), (II), (III) occur, then C has at least eight contractible edges. Therefore, we can assume that both x_0x_1 and x_5x_6 are contractible.

(a) $x_0x_1 \in R_{B'}$ and $x_5x_6 \in R_{B'}$.

Then C has exactly one non-contractible edge contradicting the above assumption.

(b) $x_0x_1 \notin R_{B'}$ and $x_5x_6 \notin R_{B'}$.

Then C has at least eight contractible edges.

(c) $x_0x_1 \in R_{B'}$ and $x_5x_6 \notin R_{B'}$.

If $|R_{B'}| \geq 4$, then C has at least eight contractible edges. Hence, assume $|R_{B'}| = 3$. Since $x'_3, x'_4 \in \overline{B}$, $x'_1x'_2$ is non-contractible and x_0x_1 is contractible, only the following two cases are possible.

(i) $x'_1 = x_2$, $x'_2 = x_1$, $x'_3 = x_0$. Since C has more than one non-contractible edges, $x'_4 \neq x_6$ and $x'_5 \neq x_6$. If $x'_5x'_6$ is contractible, then C has at least eight contractible edges. If $x'_5x'_6$ is non-contractible, then we have the case (III) for B'.

(ii) $x'_5 = x_1$, $x'_4 = x_0$, $x'_6 = x_2$. If $x'_1x'_0$ is contractible, then we have the case (III) for B'. If $x'_1x'_0$ is non-contractible, then we have the case (I) for B'.

(d) $x_0x_1 \notin R_{B'}$ and $x_5x_6 \in R_{B'}$.

If $|R_{B'}| \geq 4$, then C has at least eight contractible edges. Hence, assume $|R_{B'}| = 3$. Since $x'_3, x'_4 \in \overline{B}$, $x'_1x'_2$ is non-contractible and x_5x_6 is contractible, we must have $x'_5 = x_5, x'_4 = x_6$. Since C has more than one non-contractible edges, $x'_3 \neq x_0$ and $x'_2 \neq x_0$. We divide into two cases.

(i) $x'_1 \neq x_0$. If $x'_1x'_0$ is contractible, then C has at least eight contractible edges. If $x'_1x'_0$ is non-contractible, then we have the case (II) for B'.

(ii) $x_1' = x_0$. Let Q be an x_2'-$\{x_1, x_2\}$ path in $G - x_1' - x_5'$. Note that $Q \cap B = Q \cap B' = \emptyset$. If $x_1 \in Q$, then $x_2 \notin Q$ and $x_1x_2x_3x_4x_5x_4'x_3'a'x_1'x_2'Qx_1$ is a longer cycle. If $x_2 \in Q$, then $x_1 \notin Q$ and $x_2x_1x_4x_3ax_5a'x_3'x_4'x_1'x_2'Qx_2$ is a longer cycle.

This completes the proof of the theorem. □

The lower bound of seven in Theorem 12 is best possible, as demonstrated by the family of graphs in Figure 1a. Now, we are ready to characterize all triangle-free 3-connected graphs having a longest cycle that contains exactly six/seven contractible edges.

Theorem 13. *Let G be a triangle-free 3-connected graph and C be a longest cycle in G. Then $|E(C)| \neq 3, 4, 5, 7$, and $|E(C)| = 6$ if and only if $G \cong K_{3,k}$ ($k \geq 3$).*

Proof. Obviously, $|E(C)| \neq 3$ as G is triangle-free, and $|E(C)| = 6$ if $G \cong K_{3,k}$ ($k \geq 3$).

Suppose $|E(C)| = 4$. Let $C := x_1x_2x_3x_4x_1$. Since G is 3-connected and triangle-free, there exists a vertex x in $V(G) \setminus V(C)$. Consider any x-C 3-fan F. Then we can use F to construct a longer cycle, which is a contradiction.

Suppose $|E(C)| = 5$. Let $C := x_1x_2x_3x_4x_5x_1$. Since G is 3-connected and triangle-free, there exists a vertex x in $V(G) \setminus V(C)$. Consider any x-C 3-fan F. Then we can use F to construct a longer cycle, which is a contradiction.

Suppose $|E(C)| = 6$. Let $C := x_1x_2x_3x_4x_5x_6x_1$. Suppose $V(G) = V(C)$. Since G is 3-connected and triangle-free, $x_1x_4, x_2x_5, x_3x_6 \in E(G)$ and $G \cong K_{3,3}$. Now, let $x \in V(G) \setminus V(C)$. Consider an x-C 3-fan F_x. Since C is a longest cycle, either F_x consists of three edges xx_1, xx_3, xx_5 or F_x consists of three edges xx_2, xx_4, xx_6. Without loss of generality, assume the former. If $V(G) = V(C) \cup x$, then $x_1x_4, x_2x_5, x_3x_6 \in E(G)$ and $G \cong K_{3,4}$. Now, let y be any vertex in $V(G) \setminus (V(C) \cup x)$. Consider a y-$(C \cup x)$ 3-fan F_y. If $x \in F_y$, then $x_1, x_2, x_3, x_5, x_6 \notin F_y$ for otherwise we can construct a longer cycle. This is impossible. Therefore, $x \notin F_y$ and F_y is a y-C 3-fan. Suppose F_y consists of the three edges yx_2, yx_4, yx_6. Then $x_1xx_3x_2yx_4x_5x_6x_1$ is a longer cycle, which is a contradiction. Hence, F_y consists of the three edges yx_1, yx_3, yx_5. Note that $G - V(C)$ is independent and $x_1x_4, x_2x_5, x_3x_6 \in E(G)$. Therefore, $G \cong K_{3,k}$ ($k \geq 5$).

Suppose $|E(C)| = 7$. Let $C := x_1x_2x_3x_4x_5x_6x_7x_1$. Suppose $V(G) = V(C)$. Without loss of generality, assume $x_1x_4 \in E(G)$. Then $x_1x_5, x_4x_7 \notin E(G)$. Now, $x_5x_2, x_7x_3 \in E(G)$. Since G is triangle-free, this implies x_6 has degree two, which is impossible. Now, let $x \in V(G) \setminus V(C)$. Consider a x-C 3-fan F_x. Since C is a longest cycle, without loss of generality, assume F_x consists of three edges xx_1, xx_3, xx_6. If $x_2x_5 \in E(G)$, then $x_1x_2x_5x_4x_3xx_6x_7x_1$ is a longer cycle. Therefore, $x_5x_2 \notin E(G)$ and by symmetry $x_4x_7 \notin E(G)$. Suppose $V(G) = V(C) \cup x$. Since G is 3-connected, $x_5x_1, x_4x_1 \in E(G)$. But then, $x_1x_4x_5$ is a triangle, which is impossible. Now, let y be any vertex in $V(G) \setminus (V(C) \cup x)$ and F_y be a y-$(C \cup x)$ 3-fan. If $x \in F_y$, then $x_2, x_3, x_6, x_7 \notin F_y$ for otherwise we can construct a longer cycle. Hence, $x_4, x_5 \in F_y$. Since C is a longest cycle, F_y consists of three edges yx, yx_4, yx_5. This is impossible as yx_4x_5 is a triangle. Therefore, $x \notin F_y$ and F_y is a y-C 3-fan consisting of three y-C edges. Suppose $x_2 \in F_y$. Then $x_1, x_3 \notin F_y$. If $x_4 \in F_y$, then $x_1xx_3x_2yx_4x_5x_6x_7x_1$ is a longer cycle. If $x_5 \in F_y$, then $x_1x_2yx_5x_4x_3xx_6x_7x_1$ is a longer cycle. We have $x_6, x_7 \in F_y$, which is impossible, since yx_6x_7 is a triangle. Therefore, $x_2 \notin F_y$ and by symmetry, $x_7 \notin F_y$. We must have $x_1 \in F_y$ for otherwise we can form a longer cycle using F_y. If $x_4 \in F_y$, then $x_1x_2x_3xx_6x_5x_4yx_1$ is a longer cycle. Hence, $x_4 \notin F_y$ and by symmetry, $x_5 \notin F_y$. Therefore, F_y consists of three edges yx_1, yx_3, yx_6. Note that $G - C$ is independent. If $x_5x_2 \in E(G)$, then $x_1x_2x_5x_4x_3xx_6x_7x_1$ is a longer cycle. Therefore, $x_5x_2 \notin E(G)$ and by symmetry, $x_4x_7 \notin E(G)$. Since G is 3-connected, $x_5x_1, x_4x_1 \in E(G)$. But then, $x_1x_4x_5$ is a triangle, which is impossible. $\square$

Theorem 14. *Let G be a triangle-free 3-connected graph. Then G has a longest cycle containing exactly six contractible edges if and only if $G \cong K_{3,k}$ $(k \geq 3)$.*

Proof. If $G \cong K_{3,k}$, then all edges are contractible and every longest cycle contains exactly six edges. Suppose G has a longest cycle C containing exactly six contractible edges. By Theorem 12, C does not contain any non-contractible edges. Therefore, $|E(C)| = 6$ and $G \cong K_{3,k}$ by Theorem 13. $\square$

Theorem 15. *Let G be a triangle-free 3-connected graph. Then G has a longest cycle containing exactly seven contractible edges if and only if $G \cong G_{3,3,p,q}$ or $G \cong G_{3,3,p,q} - b_1^2a_2^2$ (see Figure 1a).*

Proof. As shown in Lemma 8, both $G_{3,3,p,q}$ and $G_{3,3,p,q} - b_1^2a_2^2$ have a longest cycle containing exactly seven contractible edges.

Suppose G has a longest cycle C containing exactly seven contractible edges. By Theorem 13, $|E(C)| \geq 8$ and thus C contains at least one non-contractible edge. By Theorem 12, C has exactly one non-contractible edge. Therefore, $|E(C)| = 8$.

Let $C := x_1x_2 \dots x_8x_1$ and define $\mathfrak{S} := \{V(e) : e \in E(C)\}$. For any $\mathfrak{S}$-end B, define $R_B := E(C) \cap (E_G(B, T_B) \cup E(B))$. By Lemmas 2 and 5, R_B contains at least three edges, all of which are contractible by Lemma 4.

Suppose for every $\mathfrak{S}$-end B, $|R_B| \geq 4$. Then C has at least eight contractible edges. Therefore, assume that there is an $\mathfrak{S}$-end B such that $|R_B| = 3$. Let B' be an $\mathfrak{S}$-end in $\overline{B}$. We will use the notation described in Lemma 6 except x_8 instead of x_0 (see also the proof of Theorem 12) .

(I) $|R_{B'}| = 3$. Then $x'_1 = x_2$, $x'_2 = x_1$, $x'_3 = x_8$, $x'_4 = x_7$, $x'_5 = x_6$. But $x_1x_2a'x_8x_7x_6x_5x_4x_3ax_1$ is a longer cycle.

(II) $|R_{B'}| = 4$. Note that $V(B') = \{x_6, x_7, x_8\}$.

(a) $\overline{B} \setminus B' = \emptyset$. Since G is 3-connected and triangle-free, $x_7x_2, x_8x_5 \in E(G)$. This implies $x_6x_1 \in E(G)$, and $G \cong G_{3,3,0,q}$ or $G \cong G_{3,3,0,q} - b_1^2a_2^2$.

(b) $\overline{B} \setminus B' \neq \emptyset$. Consider any vertex $b \in \overline{B} \setminus B'$. Let F be any b-$\{x_2, x_1, x_8, x_7, x_6, x_5\}$ 3-fan. Since C is a longest cycle, there are four possibilities: (i) $x_2, x_8, x_6 \in F$, (ii) $x_2, x_8, x_5 \in F$, (iii) $x_2, x_7, x_5 \in F$ and (iv) $x_1, x_7, x_5 \in F$. For (i) and (ii), denote the b-x_2 path in F by P_2 and the b-x_8 path in F by P_8.

Then $x_1x_2P_2bP_8x_8x_7x_6x_5x_4x_3ax_1$ is a longer cycle. For (iii), denote the b-x_2 path in F by P_2 and the b-x_7 path in F by P_7. Then $x_1x_2P_2bP_7x_7x_6x_5x_4x_3ax_1$ is a longer cycle. Therefore, only (iv) is possible, and since C is a longest cycle, F consists of the three edges bx_1, bx_7, bx_5. Since G is triangle-free, $\overline{B} - B'$ is independent. Now, $x_8x_5 \in E(G)$. If $x_6x_2 \in E(G)$, then $x_1ax_3x_4x_5x_8x_7x_6x_2x_1$ is a longer cycle. Hence, $x_6x_1 \in E(G)$. Since $G - x_1 - x_5$ is connected, $x_7x_2 \in E(G)$. Therefore, $G \cong G_{3,3,p,q}$ or $G \cong G_{3,3,p,q} - b_1^2a_2^2$ where $p \geq 1$. □

6 3-Connected Graphs of Girth at least 5

The previous section gives us a lower bound of $\min\{|E(C)|, 7\}$ for the number of contractible edges in a longest cycle C of any triangle-free 3-connected graph. Surprisingly, if the girth increases from 4 to 5, for any longest cycle, at least $\frac{1}{12}$ of its edges are contractible as shown by Theorem 18 below.

First, we introduce the concept of cross-free and closed separators studied by Kriesell [19]. Let $S, T \in \mathfrak{T}$. We say S *crosses* T if S intersects at least two components of $G - T$. It is easy to see that if S crosses T, then T intersects every component of $G - S$. This implies that S crosses T if and only if T crosses S, and that S crosses T if and only if S intersects every component of $G - T$. We call a subset $\mathfrak{S}$ of $\mathfrak{T}$ *cross-free* if S does not cross T for all $S, T \in \mathfrak{S}$. We say that $\mathfrak{S}$ is *closed* if, for all $S, T \in \mathfrak{S}$, there exists a component C of $G - S$ and a component D of $G - T$ such that $C \cap D \neq \emptyset$ and $\overline{C} \cap \overline{D} \neq \emptyset$.

Consider a set $\mathfrak{U}$ of subsets of $V(G)$. We say that a subset $\mathfrak{S}$ of $\mathfrak{T}$ *covers* $\mathfrak{U}$ if, for every $A \in \mathfrak{U}$, there exists an $S \in \mathfrak{S}$ such that $A \subseteq S$. $\mathfrak{S}$ *exclusively covers* $\mathfrak{U}$ if $\mathfrak{S}$ covers $\mathfrak{U}$ and $A \nsubseteq S$ for every $A \in \mathfrak{U}$ and every $S \in \mathfrak{T} \setminus \mathfrak{S}$. Under certain conditions on $\mathfrak{U}$, a closed exclusive cover contains a cross-free cover.

Lemma 16. *(Lemma 3 of [19]) Let G be a graph and let $\mathfrak{U}$ be a set of subsets of $V(G)$ such that $G[A]$ is complete for every $A \in \mathfrak{U}$. Suppose that $\mathfrak{S} \subseteq \mathfrak{T}$ is closed and exclusively covers $\mathfrak{U}$. Then there exists a cross-free subset $\mathfrak{R} \subseteq \mathfrak{S}$ such that $\mathfrak{R}$ covers $\mathfrak{U}$.*

A poset (partially ordered set) $(X, \leq)$ is called a *tree order* if it has a smallest element and two elements have a common upper bound if and only if they are comparable in X. The following theorem shows that we can construct a tree order on a cross-free subset of $\mathfrak{T}$.

Theorem 17. *(Theorem 1 of [19]) Let G be a graph and let $\mathfrak{S} \subseteq \mathfrak{T}$ be cross-free. Suppose that A is an $\mathfrak{S}$-end. Then for every $T \in \mathfrak{T}$ there exists a (unique) component $C(T)$ of $G - T$ containing A, and $S \leq T :\Leftrightarrow C(S) \subseteq C(T)$ for $S, T \in \mathfrak{S}$ defines a tree order $(\mathfrak{S}, \leq)$ with smallest element $N_G(A)$.*

Finally, we are ready for the main result of this section.

Theorem 18. *Every longest cycle C of a 3-connected graph of girth at least 5 contains at least $\frac{|E(C)|}{12}$ contractible edges.*

Proof. Let G be a 3-connected graph of girth at least 5 and Z be a longest cycle in G. Let X be the set of non-contractible edges in $E(Z)$ and let $Y := E(Z) \setminus X$. Denote $\mathfrak{U} = \{V(e) : e \in X\}$ and $\mathfrak{S}$ to be the set of all 3-separators containing an edge in X. Note that $\mathfrak{S}$ is closed by Lemma 1 of [19]. Since Z is a longest cycle, each 3-separator of G induces at most one edge in X. By Lemma 16 and by keeping exactly one 3-separator which contains e for all $e \in X$, there exists a cross-free set of 3-separators $\mathfrak{R} \subseteq \mathfrak{S}$ such that every edge e of X is covered by exactly one member of $\mathfrak{R}$, denoted by T_e.

Fix an $\mathfrak{R}$-end B and denote by C_e the component of $G - T_e$ that contains B. By applying Theorem 17 to $\mathfrak{R}$ and B, let $\leq$ be the tree order on $\mathfrak{R}$ defined by $T_e \leq T_f :\Leftrightarrow C_e \subseteq C_f$. Let Z_e be the set of edges of Z with at least one endvertex in $\overline{C_e}$. Denote by f_e the (end-)edge of Z_e incident with $V(e)$ and by g_e the edge of Z_e incident with the vertex in $T_e \setminus V(e)$. We observe that $T_e < T_f$ for all $f \in Z_e \cap X$. More specifically, we define the vertices a_e, b_e, c_e, p_e, q_e by $a_e b_e = e$, $a_e p_e = f_e$, $c_e q_e = g_e$ and $c_e \in T_e$. As the edges of Z incident with a vertex of some $\mathfrak{R}$-end are contractible by Lemma 4 and $\overline{C_e}$ always contains an $\mathfrak{R}$-end, we see that Z_e contains a non-trivial subpath of contractible edges.

Claim 1. For $e, f, g \in X$ with $T_e \leq T_f \leq T_g$, we have $T_e \cap T_g \subseteq T_f$.

Proof. If $T_e \cap T_g = \emptyset$, then the result follows. Let $x \in T_e \cap T_g$. Since x has a neighbor in $C_e \subseteq C_f$ and a neighbor in $\overline{C_g} \subseteq \overline{C_f}$, $x \in T_f$. $\square$

Claim 2. For $e \in X$ with $f := f_e \in X$ and $g \in Z_e \cap X$ such that $c_e \in T_g$, consider an $h \in X$ with $T_e \leq T_h \leq T_f, T_g$. Then $h \in \{e, f, g_e\}$.

Proof. By Claim 1, $a_e \in T_e \cap T_f \subseteq T_h$ and $c_e \in T_e \cap T_g \subseteq T_h$. If $b_e \in T_h$, then $h = e$. Otherwise, h must be an edge such that one endvertex is one of a_e, c_e and the other one is in $\overline{C_e}$; that is, $h \in \{f_e, g_e\}$. $\square$

Claim 3. Suppose $f := f_e \in X$. Then $T_f \neq \{a_e, p_e, c_e\}$.

Proof. Suppose $T_f = \{a_e, p_e, c_e\}$. Note that $T_e \cap T_f = \{a_e, c_e\}$, $T_e \cap C_f = \{b_e\}$, $T_f \cap \overline{C_e} = \{p_e\}$ and $T_e \cap \overline{C_f} = T_f \cap C_e = \emptyset$. Suppose $D := \overline{C_e} \cap C_f = \emptyset$. Since b_e is adjacent to a vertex in $\overline{C_e}$, $b_e p_e$ is an edge. But a_e, p_e, b_e form a triangle, a contradiction. Hence, $D \neq \emptyset$. Since Z is a longest cycle, $Z \cap (C_e \cap C_f) \neq \emptyset$ and $Z \cap (\overline{C_e} \cap \overline{C_f}) \neq \emptyset$. Therefore, $Z \cap D = \emptyset$. Take any $x \in D$. Consider an x-$\{a_e, b_e, c_e, p_e\}$ 3-fan F. We have $F \cap \{a_e, b_e, c_e, p_e\} = \{b_e, c_e, p_e\}$, for otherwise, we can use F to construct a longer cycle than Z. Since Z is a longest cycle, the x-b_e path and x-p_e path in F are both edges. But then $x b_e a_e p_e$ is a 4-cycle, a contradiction. $\square$

Claim 4. Suppose that e, $f := f_e$ and $g := g_e$ are contained in X, and T_f and T_g are comparable according to $\leq$. Then $T_g \leq T_f$, $T_g = \{c_e, q_e, a_e\}$, $\overline{C_g} = \overline{C_e} \setminus \{q_e\}$, $b_e q_e \in E(G)$ and $a_e c_e \notin E(G)$.

Proof. Suppose, on the contrary, that $T_f \leq T_g$. Since $T_e \leq T_f$, by Claim 1, $c_e \in T_e \cap T_g \subseteq T_f$. Therefore, $T_f = \{a_e, p_e, c_e\}$, which is impossible by Claim 3. This proves $T_g \leq T_f$. By Claim 1, $a_e \in T_e \cap T_f \subseteq T_g$. Therefore, $T_g = \{c_e, q_e, a_e\}$. Note that $T_e \cap T_g = \{a_e, c_e\}$, $T_e \cap C_g = \{b_e\}$ and $\overline{C_e} \cap T_g = \{q_e\}$. Denote $D := \overline{C_e} \cap C_g$. Since Z is a longest cycle, $Z \cap (C_e \cap C_g) \neq \emptyset$ and $Z \cap (\overline{C_e} \cap \overline{C_g}) \neq \emptyset$. Hence, $Z \cap D = \emptyset$. Suppose $D \neq \emptyset$ and take any $x \in D$. Consider an x-$\{a_e, b_e, c_e, q_e\}$ 3-fan F. Then we can use F to construct a longer cycle than Z, a contradiction. Therefore, $D = \emptyset$ which implies $\overline{C_g} = \overline{C_e} \setminus \{q_e\}$, $b_e q_e \in E(G)$ and $a_e c_e \notin E(G)$. □

Denote W to be the set of edges e in X for which T_e *branches*, that is, T_e has more than one upper neighbor in the tree order. Observe that a member of a tree order branches if (and only if) it is the infimum of two incomparable members. Since the number of $\mathfrak{R}$-ends distinct from the fixed end B is at least $|W| + 1$, there exists an injection α from W to these ends. For every $e \in W$, we choose an edge $\beta(e)$ from Z with at least one endvertex in $\alpha(e)$, and as any such edge is contractible by Lemma 4, this produces an injection $\beta : W \to Y$.

Claim 5. Consider two distinct $e, e' \in X$ such that $g_e = g_{e'}$ with $f := f_e \in X$ and $T_{e'} \not\leq T_e$. Then $T_e \leq T_{e'}$ and T_e branches.

Proof. Since $q_e = q_{e'} \in \overline{C_e} \cap \overline{C_{e'}}$, we have $T_e \cap \overline{C_{e'}} \neq \emptyset$ or $T_{e'} \cap \overline{C_e} \neq \emptyset$. As T_e and $T_{e'}$ are cross-free, $T_e \cap C_{e'} = \emptyset$ or $T_{e'} \cap C_e = \emptyset$ implying that $C_{e'} \subseteq C_e$ or $C_e \subseteq C_{e'}$ as C_e and $C_{e'}$ are connected. Hence, T_e and $T_{e'}$ are comparable, and $T_e \leq T_{e'}$. This implies $e' \in Z_e \cap X$. By applying Claim 2 with $g = e'$, we see that for all $h \in X$ with $T_e \leq T_h \leq T_f, T_{e'}$, h is one of e, f, g_e. Now, h cannot be g_e for otherwise $T_{g_e} = T_{g_{e'}} > T_{e'}$. So, h is one of e, f. Suppose T_f and $T_{e'}$ are comparable. If $T_{e'} \leq T_f$, then by taking $h = e'$, we have $e' = f$. We conclude that $T_f \leq T_{e'}$. By Claim 1, $c_e = c_{e'} \in T_e \cap T_{e'} \subseteq T_f$. But then $T_f = \{a_e, p_e, c_e\}$, which is impossible by Claim 3. Therefore, T_f and $T_{e'}$ are incomparable. The infimum T_h of T_f and $T_{e'}$ (which is at least T_e) is in fact equal to T_e. Hence, T_e branches. □

We now describe a set of conditions on the edges around some T_e, which implies the existence of a short cycle. Let $e \in X$ such that $f := f_e, g := g_e \in X$ and T_e does not branch. By Claim 2, the infimum of T_f, T_g is one of T_e, T_f, T_g, and thus T_f, T_g are comparable. By Claim 4, $T_g \leq T_f$, $T_g = \{c_e, q_e, a_e\}$, $\overline{C_g} = \overline{C_e} \setminus \{q_e\}$, and $b_e q_e \in E(G)$. We then advance by considering g instead of e. A new edge $h := f_g$ comes into play, whereas g_g equals to f.

Let us assume that $h \in X$ and T_g does not branch. By Claim 2, the infimum of T_h, T_f is one of T_g, T_h, T_f, and thus T_h, T_f are comparable. By Claim 4, $T_f \leq T_h$, $T_f = \{a_e, p_e, q_e\}$, $\overline{C_f} = \overline{C_g} \setminus \{q_g = p_e\}$, and $b_g q_g = c_e p_e \in E(G)$. This produces a short cycle, $c_e p_e a_e b_e q_e$, unfortunately not short enough. But we can try to advance once more by considering f instead of g, e. A new edge $i := f_f$ shows up, whereas $g_f = h$. Let us assume that $i \in X$ and T_f does not branch. As above, we have $\overline{C_h} = \overline{C_f} \setminus \{q_f\}$, and $b_f q_f = a_e q_f \in E(G)$. Now, we obtain a short cycle $b_e a_e q_f q_e$, contradicting the girth assumption.

What are the assumptions that lead to this contradiction? We have assumed that e and its two successors f, i in Z_e belong to X, and g and its successor h in Z_e belong to X. We also assumed that T_e, T_f, T_g do not branch. Therefore, one of these assumptions must fail. The idea is to map the non-contractible edge e to a contractible edge among the four edges f, g, h, i named above (if any), and to map it to a contractible edge with endvertices in $\alpha(e), \alpha(f), \alpha(g)$ if T_e, T_f, T_g, respectively, branch. The problem is to keep the preimages of this mapping φ small.

First, we order the edges of X linearly by an order $\leq'$ such that we get a depth first search order for the corresponding separators in the tree order $\leq$ (i.e. all predecessors of some T_e in the tree order are predecessors of e with respect to $\leq'$). Then we assign $\varphi(e)$ step by step according to $\leq'$ and ensure that when assigning e to $\varphi(e)$, all e' with $T_{e'} \leq T_e$ have been assigned before. Here comes the assignment rules. The objects are defined as before. In every step, we list the preconditions where earlier subrules (of higher priority) are not applicable.

(i) $e \in X$; if $f \in Y$, set $\varphi(e) := f$.

(ii) $e, f \in X$; if $e \in W$, set $\varphi(e) := \beta(e)$.

(iii) $e, f \in X$; T_e does not branch; if $g \in Y$, set $\varphi(e) := g$.

(iv) $e, f, g \in X$; T_e does not branch; if $g \in W$, set $\varphi(e) := \beta(g)$.

(v) $e, f, g \in X$; T_e, T_g do not branch; if $h \in Y$, set $\varphi(e) := h$.

(vi) $e, f, g, h \in X$; T_e, T_g do not branch; if $f \in W$, set $\varphi(e) := \beta(f)$.

(vii) $e, f, g, h \in X$; T_e, T_g, T_f do not branch. Then $i \in Y$, set $\varphi(e) := i$.

The *type* of e is the subrule among (i) to (vii) that actually applies to e. Obviously, φ maps from X to Y. We carry out a moderately simple analysis to show that the cardinalities of the preimages of φ are bounded above by a constant.

Let us call $e \in X$ *dependent* on $j \in Z$ if $j = g_e$ and e is of type (iii), (iv) or (v). If $g_e = g_{e'}$ for some $T_{e'} < T_e$ and $f_{e'} \in X$, then $T_{e'}$ branches by Claim 5 (with swapped roles of e, e'). This implies that e' is of type (i) or (ii), and $\varphi(e')$ is one of $f_{e'}, \beta(e')$. Hence, for every $j \in Z$, there is at most one edge $e \in X$ dependent on j.

Now, fix an arbitrary $k \in Y$ and look at an $e \in X$ with $\varphi(e) = k$. If e is of type (i) or (vii), then e is within distance one from k (which applies to only four edges e on Z). If e is of type (iii) or (v), then k is equal or adjacent to an edge j where e is the unique edge dependent on j, which is possible for at most three edges e on Z. If e is of type (ii) or (vi), then the unique preimage of $\varphi(e)$ under β is either equal or adjacent to e (which applies to only three edges e on Z). Finally, if e is of type (iv), then the unique preimage of $\varphi(e)$ under β is equal to j where e is the unique edge dependent on j, which is possible for at most one edge e on Z. Thus, $|\varphi^{-1}(k)| \leq 4 + 3 + 3 + 1 = 11$ implying $|X| \leq 11|Y|$. Therefore, $|Y| \geq \frac{|Z|}{12}$ and the proof of the theorem is complete. □

7 Summary

We gather all the results of this paper on $f(k)$ and conclude with some open problems.

For all $n \geq 5$, consider the family of graphs D_n, which is constructed from the square of paths together with an extra vertex x.

$$\begin{aligned} V(D_n) :=& \{x, x_1, x_2, \ldots, x_n\} \\ E(D_n) :=& \{x_i x_{i+1} : 1 \leq i \leq n-1\} \cup \\ & \{x_i x_{i+2} : 1 \leq i \leq n-2\} \cup \\ & \{x_1 x_4, x_{n-3} x_n, x x_1, x x_2, x x_{n-1}, x x_n\} \end{aligned}$$

It is easy to see that $x x_1 x_2 \ldots x_n x$ is a Hamiltonian cycle that contains exactly six contractible edges $x x_1, x_1 x_2, x_2 x_3, x_{n-2} x_{n-1}, x_{n-1} x_n, x_n x$. The bound in Theorem 7 is therefore tight for 3-connected graphs that have minimum degree at least 4.

	triangle-free	$\delta(G) \geq \frac{3}{2}k - 1$
$k = 3$	$f(k) = 7$ ▷ Thms. 12 and 15	$f(k) = 6$ ▷ Thm. 7 and D_n
$k \geq 4$	$f(k) \geq 6$ ▷ Thm. 7	$f(k) \geq 6$ ▷ Thm. 7 except $G_{4,k}$ and $G_{5,k}$
	$f(k) \leq 2k + 2$ ▷ Lem. 9	$f(k) \leq 2k + 2$ ▷ Lem. 11

Table 1: Results on $f(k)$.

Problem 1. Characterize all 3-connected graphs with minimum degree at least 4 that contain a longest cycle with exactly six contractible edges.

Problem 2. For every $k \geq 4$, find all k-connected graphs with minimum degree at least $\frac{3}{2}k - 1$ that contain a longest cycle with less than eight contractible edges.

Problem 3. Improve the lower and upper bounds for $f(k)$.

Problem 4. For 3-connected graphs of girth at least 5, determine the supremum for $k \in \mathbb{R}$ such that every longest cycle C contains at least $k|E(C)|$ contractible edges.

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