More on foxes

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Abstract

An edge in a k-connected graph G is called *k-contractible* if the graph G/e obtained from G by contracting e is k-connected. Generalizing earlier results on 3-contractible edges in spanning trees of 3-connected graphs, we prove that (except for the graphs K_{k+1} if $k \in \{1,2\}$) (a) every spanning tree of a k-connected triangle free graph has two k-contractible edges, (b) every spanning tree of a k-connected graph of minimum degree at least $\frac{3}{2}k-1$ has two k-contractible edges, (c) for $k>3$, every DFS tree of a k-connected graph of minimum degree at least $\frac{3}{2}k-\frac{3}{2}$ has two k-contractible edges, (d) every spanning tree of a cubic 3-connected graph nonisomorphic to K_4 has at least $\frac{1}{3}|V(G)|-1$ many 3-contractible edges, and (e) every DFS tree of a 3-connected graph nonisomorphic to K_4, the prism, or the prism plus a single edge has two 3-contractible edges. We also discuss in which sense these theorems are best possible.

AMS classification: 05c40, 05c05.

Keywords: Contractible edge, spanning tree, DFS tree, fox.

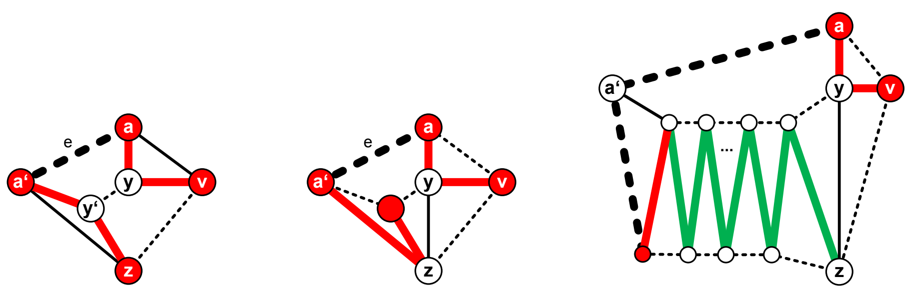
1 Introduction

All graphs throughout are assumed to be finite, simple, and undirected. For terminology not defined here we refer to [2] or [1]. A graph is called *k-connected* ($k \geq 1$) if $|V(G)| > k$ and $G-T$ is connected for all $T \subseteq V(G)$ with $|T| < k$. Let $\kappa(G)$ denote the *connectivity* of G, that is, the largest k such that G is k-connected. A set $T \subseteq V(G)$ is called a *smallest separating set* if $|T| = \kappa(G)$ and $G-T$ is disconnected. By $\mathfrak{T}(G)$ we denote the set of all smallest separating sets of G. An edge e of a k-connected graph G is called *k-contractible* if the graph G/e obtained from G by *contracting* e, that is, identifying its endvertices and simplifying the result, is k-connected. No edge in K_{k+1} is k-contractible, whereas all edges in K_ℓ are if $\ell \geq k+2$, and it is well-known and straightforward to check that, for a noncomplete k-connected graph G, an edge e is not k-contractible if and only if $\kappa(G)=k$ and $V(e) \subseteq T$ for some $\mathfrak{T}(G)$.

There is a rich literature dealing with the distribution of k-contractible edges in k-connected graphs (see the surveys [6, 5]), with a certain emphasis on the case $k = 3$. In [4], 3-connected graphs that admit a spanning tree without any 3-contractible edge have been introduced; these were called *foxes* (see Figure 1). For example, every wheel G is a fox, which is certified by the spanning star Q that is centered at the hub of the wheel. However, Q is as far from being a *DFS* (depth-first search) tree as it can be, and one could ask if the property of being a fox can be certified by some DFS tree at all. The answer is no, as it has been shown in [4] that every DFS tree of every 3-connected graph nonisomorphic to K_4 does contain a 3-contractible edge. Here we generalize the latter result as follows.

Theorem 1 *Every DFS tree of every 3-connected graph nonisomorphic to K_4, the prism $K_3 \times K_2$, or the unique graph $(K_3 \times K_2)^+$ obtained from $K_3 \times K_2$ by adding a single edge contains at least two 3-contractible edges.*

Theorem 1 is best possible in the sense that there is an infinite class of 3-connected graphs admitting a DFS tree with only two 3-contractible edges (see Figure 1c).



(a) The prism $K_3 \times K_2$, which is no fox. Dashed edges are 3-contractible. Fat edges depict a DFS tree that contains exactly one 3-contractible edge (namely, e).

(b) The fox $(K_3 \times K_2)^+$ and a DFS-tree of it containing exactly one 3-contractible edge (namely, e).

(c) An infinite family of foxes obtained by enlarging the lower left part horizontally. Every fox in this family has a DFS tree (fat edges) that contains exactly two 3-contractible edges.

Figure 1: The two exceptional graphs of Theorem 1 on six vertices and an infinite family of foxes showing that Theorem 1 is sharp.

Our proofs are based on methods introduced by Mader in [8], generalizing the concept of critical connectivity. This approach makes it possible to generalize some of the earlier results on foxes from 3-connected graphs to certain classes of k-connected graphs.

Extending the definition above, let us define a *k-fox* to be a k-connected graph admitting a spanning tree without k-contractible edges. For $k \geq 4$, there are graphs G without k-contractible edges at all, and every such G is, trivially, a k-fox; thus, the question is interesting only under additional constraints to G

which force k-contractible edges. Classic constraints are to forbid triangles or to bound the vertex degrees from below: In [10] it has been proven that every triangle free k-connected graph contains a k-contractible edge, and in [3], it has been shown that every k-connected graph of minimum degree at least $\frac{5k-3}{4}$ must contain a k-contractible edge (unless G is isomorphic to K_{k+1} when $k \leq 3$). These results do have a common root in terms of generalized criticity [8], and so it is perhaps not surprising that the following new result, Theorem 2, follows from a statement on special separating sets (Theorem 7 in Section 2).

Theorem 2 *Let G be a k-connected graph (except for K_{k+1} if $k \in \{1, 2\}$) that is triangle free or of minimum degree at least $\frac{3}{2}k - 1$. Then every spanning tree of G contains at least two k-contractible edges.*

This implies that k-foxes must contain triangles as well as vertices of "small" degree. In order to show that the bound in Theorem 2 is best possible, we exhibit k-connected graphs of minimum degree $\frac{3}{2}k - \frac{3}{2}$ (and necessarily containing triangles) that admit a spanning tree with no k-contractible edge. For odd $k \geq 3$, take the lexicographic product of any cycle and $K_{(k-1)/2}$ and add an additional vertex plus all edges connecting it to the others. (So for $k = 3$ we get back the wheels.) The resulting graph is k-connected and of minimum degree $\frac{3}{2}k - \frac{3}{2}$, and the spanning star centered at the additional vertex has no k-contractible edge. The same construction works, more generally, if instead of a cycle we start with any *critically* 2*-connected graph*, that is, a 2-connected graph G such that for every vertex x the graph $G - x$ is not 2-connected. However, for DFS trees the situation changes once more:

Theorem 3 *For $k > 3$, every DFS tree of every k-connected graph of minimum degree at least $\frac{3}{2}k - \frac{3}{2}$ contains at least two k-contractible edges.*

Observe that the statement of Theorem 3 remains true for $k = 3$ by Theorem 1 unless the graph is one of the three exceptions listed there.

Theorem 2 provides a particularly simple proof that every spanning tree of a *cubic* 3-connected graph nonisomorphic to K_4 or the prism has at least two 3-contractible edges (see Corollary 2 in Section 2); however, taking more external knowledge into account we can improve *two* to the following sharp linear bound in terms of $|V(G)|$ (end of Section 2).

Theorem 4 *Every spanning tree of every cubic 3-connected graph nonisomorphic to K_4 contains at least $\frac{1}{3}|V(G)| - 1$ many 3-contractible edges. The bound is sharp, also when restricted to DFS trees.*

We also show sharpness for Theorem 4. Obtain a graph G' from any cubic 3-connected graph G by replacing every vertex x with a triangle Δ_x such that, for every incident edge e of x, the end vertex x of e is replaced with a unique vertex of Δ_x. Clearly, G' is cubic and 3-connected. Let T be a spanning tree of G, and let T' be formed by all edges of T together with the edges of a spanning path of each Δ_x. Then T' is a spanning tree of G' with exactly $\frac{1}{3}|V(G')| - 1$ many 3-contractible edges, as no edge in a triangle is 3-contractible. When restricted to DFS trees, assume in addition that G is Hamiltonian and let T be a Hamiltonian path of G. Then the paths of each Δ_x can be chosen such that T' is a Hamiltonian path of G' and we see that there is no improvement for DFS trees in general.

2 Contractible edges in spanning trees

Let G be a graph and $\mathfrak{T}(G)$ be the set of its smallest separating sets. For $T \in \mathfrak{T}(G)$, the union of the vertex sets of at least one but not of all components of $G - T$ is called a *T-fragment*. Obviously, if F is a T-fragment then so is $\overline{F}^G := V(G) \setminus (T \cup F)$, where the index G is always omitted as it will be clear from the context. Moreover, $\overline{\overline{F}} = F$. Fragments have the following fundamental property.

Lemma 1 [8] *Let B be a T_B-fragment and F be a T-fragment of a graph G such that $B \cap F \neq \emptyset$. Then $|B \cap T| \geq |\overline{F} \cap T_B|$, and if equality holds then $B \cap F$ is a $(B \cap T) \cup (F \cap T_B) \cup (T \cap T_B)$-fragment.*

Proof. Let $k := \kappa(G)$ and observe that $N_G(B \cap F)$ separates G and is a subset of $X := (B \cap T) \cup (F \cap T_B) \cup (T \cap T_B)$. Therefore, $k \leq |N_G(B \cap F)| \leq |X| = |B \cap T| + |T_B| - |T_B \cap \overline{F}| = |B \cap T| + k - |T_B \cap \overline{F}|$. Since k cancels on both sides, rearranging the terms yields the desired inequality, and equality implies $N_G(B \cap F) = X$. □

We will not give explicit references to Lemma 1, but mark estimations or conclusions based on it by $\star$; for example, we write $|F \cap T'| \geq^\star |F' \cap T|$ if F is a T-fragment and F' is a T'-fragment such that $F \cap \overline{F'} \neq \emptyset$ to indicate that the inequality is a straightforward application of Lemma 1. This convention also applies to the following slightly more complex but standard application of Lemma 1: If both $B \cap F$ and $\overline{B} \cap \overline{F}$ are nonempty, then, by Lemma 1, they are both fragments. In many cases, B will be an inclusion minimal fragment with respect to some property, F will be a T-fragment such that T contains a vertex from B, and $F \cap B \neq \emptyset$ will have the same property as B (but is no fragment by minimality): In such a scenario, we infer $|B \cap T| \geq |\overline{F} \cap T_B| + 1$, $|F \cap T_B| \geq |\overline{B} \cap T| + 1$, and $\overline{F} \cap \overline{B} = \emptyset$ from Lemma 1, and again refer to it by $\star$, for example, by writing $|B \cap T| \geq^\star |\overline{F} \cap T_B| + 1$, $|F \cap T_B| \geq^\star |\overline{B} \cap T| + 1$, or $\overline{F} \cap \overline{B} =^\star \emptyset$, respectively.

Another fact that will be used throughout is the following.

Lemma 2 *Let B be a T_B-fragment and F be a T-fragment of a graph G such that $F \subseteq T_B$. Then none of the sets $B \cap T$, $\overline{B} \cap T$ and $\overline{F} \cap T_B$ is empty.*

Proof. Since T_B is a smallest separating set, every vertex of $F \subseteq T_B$ has a neighbor in B as well as in $\overline{B}$. Since these neighbors may only be in T, $B \cap T \neq \emptyset \neq \overline{B} \cap T$. The same reasoning for T implies that every component of $\overline{F}$ is adjacent to all vertices of T. Since $B \cap T \neq \emptyset \neq \overline{B} \cap T$, every such component contains a vertex from T_B. □

Now let us fix a subset $\mathfrak{S}$ of the power set $\mathfrak{P}(V(G))$. We call a T-fragment F a *T-$\mathfrak{S}$-fragment* if $S \subseteq T$ for some $S \in \mathfrak{S}$. In that case, again, $\overline{F}$ is a T-$\mathfrak{S}$-fragment, too; F is called a *T-$\mathfrak{S}$-end* if there is no T'-$\mathfrak{S}$-fragment properly contained in it, and F is called a *T-$\mathfrak{S}$-atom* if there does not exist a T'-$\mathfrak{S}$-fragment with fewer than $|F|$ vertices. Observe that if F is a T-fragment then necessarily $T = N_G(F)$, so that T can be reconstructed from F; therefore, one might omit T in the notion, which defines the terms *fragment*, $\mathfrak{S}$-*fragment*, $\mathfrak{S}$-*end*, and $\mathfrak{S}$-*atom*. These definitions and the following theorem are from [8] and have their roots back in a 1970 paper by Watkins where it was proven that the degrees of a vertex transitive k-connected graph are at most $\frac{3}{2}k - 1$ [11].

Theorem 5 [8] *Let G be a graph, $\mathfrak{S} \subseteq \mathfrak{P}(V(G))$, and A be a T_A-$\mathfrak{S}$-atom of G. Suppose that there exists an $S \in \mathfrak{S}$ and a $T \in \mathfrak{T}(G)$ such that $S \subseteq T \setminus \overline{A}$ and $T \cap A \neq \emptyset$. Then $A \subseteq T$ and $|A| \leq |T \setminus T_A|/2$.*

A fragment of minimum size is usually called an *atom* of G. Consequently, for $\mathfrak{S} := \{\emptyset\}$, we obtain the following specialization of Theorem 5, which appeared already in [9].

Theorem 6 [9] *Let G be a graph and A be a T_A-atom of G. Suppose that $A \cap T \neq \emptyset$ for some $T \in \mathfrak{T}(G)$. Then $A \subseteq T$ and $|A| \leq |T \setminus T_A|/2 \leq \kappa(G)/2$.*

We start our considerations with the following result.

Theorem 7 *Let Q be a spanning tree of a graph G of connectivity k, set $\mathfrak{S} := \{V(e) : e \in E(Q)\}$, suppose that all $\mathfrak{S}$-fragments have cardinality at least $\frac{k-1}{2}$, and let B be an $\mathfrak{S}$-end. Then $|B| = \frac{k-1}{2}$ (in particular, k is odd) or all edges e from Q with $|V(e) \cap B| = 1$ are k-contractible.*

Proof. Let $a := \frac{k-1}{2}$ and $T_B := N_G(B)$. Observe that the existence of B implies $k > 1$. Since Q is a spanning tree, there exists an edge e with $|V(e) \cap B| = 1$. If all such edges e are k-contractible then we are done. Otherwise, one such edge e is not k-contractible; there exists a $T \in \mathfrak{T}(G)$ with $V(e) \subseteq T$, and we consider a T-fragment F. Now B and F are $\mathfrak{S}$-fragments, so that $|B|, |F|, |\overline{B}|, |\overline{F}| \geq a$, and it suffices to prove that $|B| \leq a$.

Observe that $V(e) \cap T_B \neq \emptyset$, so $|T \cap T_B| \geq 1$. If $B \cap F \neq \emptyset \neq B \cap \overline{F}$, we infer $\overline{B} \subseteq^\star T$ and $2|\overline{B}| = 2|\overline{B} \cap T| \leq^\star |F \cap T_B| - 1 + |\overline{F} \cap T_B| - 1 = |T_B \setminus T| - 2 \leq k - 3$, and, hence, $|\overline{B}| \leq (k-3)/2 < a$, which is a contradiction. Suppose that $B \cap F \neq \emptyset$. Then $B \cap \overline{F} = \emptyset$, which implies $\overline{F} \subseteq^\star T_B$. If $\overline{B} \cap F \neq \emptyset$, then $|\overline{B} \cap T| \geq^\star |\overline{F} \cap T_B| = |\overline{F}| \geq a$, and otherwise $|\overline{B} \cap T| = |\overline{B}| \geq a$, too. Hence, $k = |T| = |B \cap T| + |\overline{B} \cap T| + |T_B \cap T| \geq^\star (|\overline{F} \cap T_B| + 1) + a + 1 \geq 2a + 2 = k + 1$, which is a contradiction. Consequently, $B \cap F = \emptyset$, and, by the same argument, $B \cap \overline{F} = \emptyset$. If $\overline{B} \cap F \neq \emptyset$, we infer $|F \cap T_B| \geq^\star |B \cap T| = |B| \geq a$, and otherwise $|F \cap T_B| = |F| \geq a$, too. Symmetrically, we get $|\overline{F} \cap T_B| \geq a$ and conclude $|T \cap T_B| = 1$. Now $\overline{B} \cap F \neq \emptyset$ implies $|\overline{B} \cap T| \geq^\star |\overline{F} \cap T_B| \geq a$, $\overline{B} \cap \overline{F} \neq \emptyset$ implies $|\overline{B} \cap T| \geq^\star |F \cap T_B| \geq a$, and otherwise $\overline{B} \cap F = \emptyset = \overline{B} \cap \overline{F}$ implies $|\overline{B} \cap T| = |\overline{B}| \geq a$, too. It follows $|B| = |B \cap T| = |T| - |T \cap T_B| - |\overline{B} \cap T| \leq k - 1 - a = a$, which proves the theorem. $\square$

Corollary 1 *Let $G \ncong K_{k+1}$ be a k-connected graph in which every fragment has cardinality at least $\frac{k}{2}$. Then every spanning tree of G admits at least two k-contractible edges.*

Proof. Let Q be a spanning tree of G. Then $|E(Q)| \geq 2$, since $|V(G)| \geq k+2 \geq 3$. Hence, we may assume that at least one edge in Q is not k-contractible. Therefore, $\kappa(G) = k$, and there exists an $\mathfrak{S}$-fragment C, where we define $\mathfrak{S} := \{V(e) : e \in E(Q)\}$ as before. In particular, $k > 1$. Consider an $\mathfrak{S}$-end $B \subseteq C$. By Theorem 7, Q contains a k-contractible edge e that has precisely one end vertex in C. Likewise, consider an $\mathfrak{S}$-end $B \subseteq \overline{C}$. By applying Theorem 7 once more, Q contains a k-contractible edge f that has precisely one end vertex in $\overline{C}$. Clearly, $e \neq f$, which proves the statement. $\square$

For $k \leq 2$, the fragment size condition of Corollary 1 is trivially true, and so every spanning tree of every k-connected graph nonisomorphic to K_{k+1}, $k \leq 2$, admits at least two 2-contractible edges. For general k, Corollary 1 implies Theorem 2 as follows, and the examples beneath the latter in the introduction show also that the bound on the fragment size in Corollary 1 cannot be improved.

Proof of Theorem 2. As argued above, the statement holds for $k \leq 2$, so let $k \geq 3$. Then $G \ncong K_{k+1}$, since G is triangle free or of minimum degree at least $\frac{3}{2}k - 1$. If the minimum degree condition is satisfied, every fragment has cardinality at least $\frac{k}{2}$, and applying Corollary 1 gives the claim. If G is triangle free, let Q be a spanning tree of G and let $\mathfrak{S} := \{V(e) : e \in E(Q)\}$. As in the proof of Corollary 1, we may assume that at least one edge in Q is not k-contractible, which implies $\kappa(G) = k$ and the existence of a $\mathfrak{S}$-fragment C. Since G is triangle free, every $\mathfrak{S}$-fragment contains two adjacent vertices, and considering the neighborhood of the two vertices of degree at least k each implies that every $\mathfrak{S}$-fragment has in fact cardinality at least k. By applying Theorem 7 twice, as in the previous proof, we find two k-contractible edges $e \neq f$ in Q with end vertices in C and $\overline{C}$, respectively. $\square$

As promised in the introduction we derive the following result from Theorem 2.

Corollary 2 *Every spanning tree of every cubic* 3*-connected graph nonisomorphic to* K_4 *or the prism* $K_3 \times K_2$ *contains at least two* 3*-contractible edges.*

Proof. We use induction on the number of vertices. The induction starts for K_4, so suppose that G is a cubic 3-connected graph on at least six vertices, and Q is a spanning tree of G. We may assume that G contains a triangle Δ, for otherwise the statement follows from Theorem 2, and we may assume that G is not the prism. The edge neighborhood of any such triangle forms a matching of three 3-contractible edges, at least one of which belongs to Q; in fact, we may assume that exactly one edge of the edge neighborhood belongs to Q, for otherwise the statement is proven. So suppose that e is the only edge from Q in the edge neighborhood of Δ. If there was another triangle Δ' then, consequently, e is the only edge from Q in the edge neighborhood of Δ', too, implying that $V(G) = V(\Delta) \cup V(\Delta')$, so that G is the prism $K_3 \times K_2$, which is a contradiction. So we may assume that Δ is the only triangle in G. The graph G/Δ obtained from G by identifying the three vertices of Δ and simplifying is not K_4, as G is not the prism, and G/Δ is not the prism, as Δ is the only triangle in G and the prism has two vertex-disjoint triangles. Clearly, G/Δ is cubic and Q/Δ is a spanning tree of G. Since no smallest separating set of G contains two vertices of Δ, G/Δ is also 3-connected. Hence, by induction, Q/Δ contains two 3-contractible edges of G/Δ, and the two edges corresponding to these in G are 3-contractible in G as one checks readily. □

By using a powerful result on 3-contractible edges in 3-connected graphs from the literature we can improve Corollary 2 to Theorem 4 (where the lower bound to the number of 3-contractible edges is sharp).

Proof of Theorem 4. From Lemma 3 and Lemma 4 in [7], we get Theorem 3 in [6], which implies, together with Theorem 12 from [6], that *every vertex in a* 3*-connected graph is either contained in a triangle or on at least two* 3*-contractible edges.* We first show that this implies for any 3-connected *cubic* graph G that its subgraph H on $V(G)$ formed by the non-3-contractible edges of G is a clique factor (that is, all components of H are isolated vertices, or single edges, or triangles — or K_4 in case that G is K_4): Suppose that G is not K_4. If x has degree at least 2 in H, then x is on a triangle Δ in G by the result mentioned above and this triangle is also in H, whereas the edges from its edge neighborhood are not (so x is in a triangle component of H). Now every spanning tree Q contains at most 2 edges from every triangle in H, so that it contains at most $\frac{2}{3}|V(G)|$ edges from H. Therefore, Q contains at least $|V(G)| - 1 - \frac{2}{3}|V(G)| = \frac{1}{3}|V(G)| - 1$ 3-contractible edges (unless G is K_4). □

3 Contractible edges in DFS trees

Again, we observe that the spanning tree in the sharpness example of Corollary 1 and Theorem 2 is far from being a DFS tree. The following theorem will provide more insight into the distribution of k-contractible edges in spanning trees of graphs where the fragment lower bound $\frac{k-1}{2}$ from Theorem 7 is sharp (as opposed to the bound $\frac{k}{2}$ that is sharp in Corollary 1), and leads to a proof of Theorem 3 and, in the next section, of Theorem 1.

Theorem 8 *Let Q be a spanning tree of a graph G of connectivity k, set $\mathfrak{S} := \{V(e) : e \in E(Q)\}$, suppose that all $\mathfrak{S}$-fragments have cardinality at least $\frac{k-1}{2}$, set $\mathfrak{R} := \{V(e) : e \in E(Q), |V(e) \cap A| = 1$ for some $\mathfrak{S}$-end $A\}$, and let B be an $\mathfrak{R}$-end. Then $|B| = \frac{k-1}{2}$, or Q contains a k-contractible edge e with at least one endvertex in B, or $N_G(B)$ contains an $\mathfrak{R}$-fragment of cardinality $\frac{k-1}{2}$ such that all edges from Q having exactly one vertex in common with it are k-contractible.*

Proof. Let $a := \frac{k-1}{2}$. Let us call an edge from Q *green* if it is not k-contractible.

Claim 1. Suppose that e is a green edge and A is an $\mathfrak{S}$-end with $|V(e) \cap A| = 1$. Then $|A| = a$ (so that A is an $\mathfrak{S}$-atom, and a is an integer), and $A \subseteq T$ for every $T \in \mathfrak{T}(G)$ with $V(e) \subseteq T$ (and there exists such a T).

We get $|A| = a$ immediately from Theorem 7. Since e is green, there exists a $T \in \mathfrak{T}(G)$ such that $V(e) \subseteq T$, and for every such T we know $V(e) \subseteq T \setminus \overline{A}$ and $A \cap T \neq \emptyset$, so that $A \subseteq T$ follows from Theorem 5, proving Claim 1.

Now let us call a green edge e *red* if $|V(e) \cap A| = 1$ for some $\mathfrak{S}$-end A (see Figure 1c for an example of this coloring in the special case $k = 3$). Let $T_B := N_G(B)$. By definition of $\mathfrak{R}$ and Claim 1, there exists a red edge e' and an $\mathfrak{S}$-end A' with $|V(e') \cap A'| = 1$ and $A' \cup V(e') \subseteq T_B$. Since B is an $\mathfrak{S}$-fragment (every $\mathfrak{R}$-fragment is an $\mathfrak{S}$-fragment by definition), it must contain an $\mathfrak{S}$-end A. There exists an edge e from Q with $|V(e) \cap A| = 1$. If e is k-contractible, we are done; thus we may assume that e is green, so that e is even red.

By definition and Claim 1, there exists a $T \in \mathfrak{T}(G)$ such that $A \cup V(e) \subseteq T$. We now consider a T-fragment F, which is, in fact, an $\mathfrak{R}$-fragment, and analyze the possible ways $F, \overline{F}$ meet $B, \overline{B}$ (for example, $B = A = \{v\}$, $e' = ya$ and $F = A' = \{a\}$ in Figure 1c). We will show that regardless of whether $B \cap F$ is empty or not, $T \cap T_B = \emptyset$, $B \cap T = A \cup V(e)$ with cardinality $a+1$, $F \cap T_B = A' \cup V(e')$ with cardinality $a+1$, and $\overline{F} = \overline{F} \cap T_B$ is an $\mathfrak{R}$-atom with cardinality a and thus also an $\mathfrak{S}$-atom.

Again we may rule out that $B \cap F \neq \emptyset \neq B \cap \overline{F}$, as this would imply $\overline{B} \subseteq^\star T$ and $2|\overline{B}| = 2|\overline{B} \cap T| \leq^\star |\overline{F} \cap T_B| - 1 + |F \cap T_B| - 1 \leq k-2$, and hence $|\overline{B}| \leq \frac{k-2}{2} < a$, which gives a contradiction.

Now assume that exactly one of $B \cap F$ and $B \cap \overline{F}$ is nonempty, say, by symmetry of F and $\overline{F}$, $B \cap F \neq \emptyset$ (see Figure 2). It follows $B \cap \overline{F} = \emptyset =^\star \overline{B} \cap \overline{F}$. Consequently, $|\overline{F} \cap T_B| = |\overline{F}| \geq a$ and, hence, $|B \cap T| \geq^\star |\overline{F} \cap T_B| + 1 \geq a + 1$. If $\overline{B} \cap F \neq \emptyset$, then $|\overline{B} \cap T| \geq^\star |\overline{F} \cap T_B| \geq a$, and if otherwise $\overline{B} \cap F = \emptyset$, then $|\overline{B} \cap T| = |\overline{B}| \geq a$, too. Now $k = |T| = |B \cap T| + |\overline{B} \cap T| + |T_B \cap T| \geq (a+1) + a + 0 = k$, so we get equality summand-wise, that is, $|B \cap T| = a + 1$,

$|\overline{B} \cap T| = a$, and $T \cap T_B = \emptyset$. This implies $a \leq |\overline{F}| \leq^\star |B \cap T| - 1 = a$, so that $|\overline{F}| = a$ (in particular, $\overline{F}$ is an $\mathfrak{R}$-atom) and $B \cap T = A \cup V(e)$. Moreover, since $A' \cup V(e') \subseteq T_B$ and $|A' \cup V(e')| \geq a+1 > |\overline{F}|$, we conclude $F \cap T_B = A' \cup V(e')$, as $G[A' \cup V(e')]$ is connected.

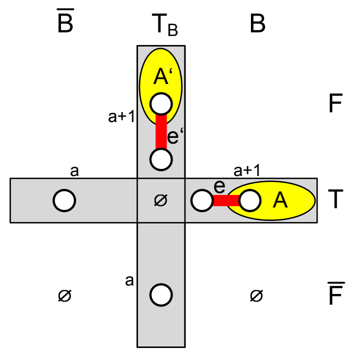


Figure 2: The structure of the $\mathfrak{R}$-fragments B and F.

In the remaining case $B \cap F = \emptyset = B \cap \overline{F}$, i.e. $B \subseteq T$, we essentially obtain the same picture (Figure 2): If $F \cap \overline{B} \neq \emptyset$, then $|F \cap T_B| \geq^\star |B \cap T| = |B| \geq a$, and if otherwise $F \cap \overline{B} = \emptyset$, then $|F \cap T_B| = |F| \geq a$, too; symmetrically, $|\overline{F} \cap T_B| \geq a$. Then $F \cap \overline{B} \neq \emptyset$ implies $|\overline{B} \cap T| \geq^\star |\overline{F} \cap T_B| \geq a$, $\overline{F} \cap \overline{B} \neq \emptyset$ implies $|\overline{B} \cap T| \geq |F \cap T_B| \geq a$, and the remaining case $F \cap \overline{B} = \emptyset = \overline{F} \cap \overline{B}$ implies $|\overline{B} \cap T| = |\overline{B}| \geq a$, too. Now if $|B| = a$, the theorem is proved, so let $|B| > a$. Then $k = |T| = |B| + |T \cap T_B| + |\overline{B} \cap T| \geq (a+1) + 0 + a = k$, so that equality holds summand-wise, implying $T \cap T_B = \emptyset$, $B = B \cap T = A \cup V(e)$, $|B| = a+1$, and $|\overline{B} \cap T| = a$. By symmetry of F and $\overline{F}$, we may assume that $A' \cup V(e') \subseteq F$ and hence $|F \cap T_B| = a+1$ and $|\overline{F} \cap T_B| = a$. This implies $\overline{B} \cap \overline{F} = \emptyset$, as otherwise $k = |F \cap T_B| + |\overline{F} \cap T_B| \geq^\star |A' \cup V(e')| + |B \cap T| = (a+1) + (a+1) > k$ gives a contradiction.

Regardless of whether $B \cap F$ is empty or not, we conclude $T \cap T_B = \emptyset$, $B \cap T = A \cup V(e)$ with cardinality $a+1$, $F \cap T_B = A' \cup V(e')$ with cardinality $a+1$, and $\overline{F} = \overline{F} \cap T_B$ is an $\mathfrak{R}$-atom with cardinality a and thus also an $\mathfrak{S}$-atom. We proceed with the general argument.

If all edges from Q that connect $\overline{F}$ to T are k-contractible, we are done. So there exists a green edge xy with $x \in \overline{F}$ and $y \in T$ and a $T' \in \mathfrak{T}(G)$ with $\{x, y\} \subseteq T'$, and by Claim 1 (applied to xy and $\overline{F}$) we obtain $\overline{F} \subseteq T'$. We discuss the possible locations of y and will show that all are impossible, which proves the theorem. Observe that A must have at least $a = |A|$ neighbors in $\overline{F}$, for otherwise $\overline{F} \setminus N_G(A)$ would be a nonempty set with less than k neighbors in G, which contradicts the fact that G is k-connected. It follows $\overline{F} \subseteq N_G(A)$.

If y would be the vertex in $V(e) \setminus A$, then xy would be a red edge with its endvertices in $N_G(A)$, certifying that A is an $\mathfrak{R}$-fragment properly contained in the $\mathfrak{R}$-end B, which is absurd. If y would be some vertex in A, then xy

would be a red edge, and by Claim 1 (applied to xy and A), we get $A \subseteq T'$. Hence, T' intersects B and contains the $a + a = k - 1$ vertices from $A \cup \overline{F}$. Since T' has k vertices, exactly one of its vertices is not in $A \cup \overline{F}$. Since B is an $\mathfrak{R}$-end, this vertex must be in $\overline{B}$, as any other choice would imply that a T'-$\mathfrak{R}$-fragment is strictly contained in B. It follows that T' and T_B *cross* (i.e. $T' \cap B \neq \emptyset \neq T' \cap \overline{B}$). Consider a T'-fragment F' that contains $T_B \cap F$ (this exists, as $G[T_B \cap F]$ is connected and does not intersect T'). Because T' and T_B cross, T_B must intersect $\overline{F'}$, which contradicts $T_B \setminus F = \overline{F} \subseteq T'$.

It follows that, necessarily, $y \in \overline{B}$. Suppose to the contrary that T' separates $A \cup V(e)$, that is, there exists a T'-fragment F' such that $F' \cap (A \cup V(e)) \neq \emptyset$ and $\overline{F'} \cap (A \cup V(e)) \neq \emptyset$. Then T' must contain a vertex z from A, and T_B and T' cross, so that T' separates $A' \cup V(e') \supseteq T_B \setminus T'$, too. Thus, T' contains a vertex z' from A'. Since T' cannot separate the end vertices of $V(e')$, we know that $V(e') \cap F' = \emptyset$ or $V(e') \cap \overline{F'} = \emptyset$. Without loss of generality we may suppose that $V(e') \cap \overline{F'} = \emptyset$. If $F' \cap \overline{B} \neq \emptyset$, then $F' \cap \overline{B}$ would be an $S := (F' \cap T_B) \cup (T' \cap T_B) \cup (T' \cap \overline{B})$-fragment, as $\overline{F'} \cap B \neq \emptyset$. Since e' is a green edge with exactly one end vertex from the $\mathfrak{S}$-atom A' and contained in S we know from Claim 1 that $A' \subseteq S$, contradiction. If $\overline{F'} \cap \overline{B} \neq \emptyset$, then $F' \cap B$ would be an $P := (F' \cap T_B) \cup (T' \cap T_B) \cup (T' \cap B)$-fragment containing $V(e')$, as $F' \cap B \neq \emptyset$. This implies again $A' \subseteq P$, contradiction. Therefore, $\overline{B} \subseteq T'$. It follows $k = |T'| = |\overline{B} \cap T'| + |T_B \cap T'| + |B \cap T'| \geq |\overline{B}| + |\overline{F} \cup \{z'\}| + |\{z\}| \geq a + (a + 1) + 1 > k$, contradiction.

Hence we have to assume that T' does not separate $A \cup V(e)$. Consequently, there exists a T'-fragment F' such that $F' \cap (A \cup V(e)) = \emptyset$. If $F' \cap F \neq \emptyset$, then $|F' \cap T| \geq^\star |\overline{F} \cap T'| = |\overline{F}| \geq a$, and if otherwise $F' \cap F = \emptyset$, then $|F' \cap T| = |F'| \geq a$, too. On the other hand, $|F' \cap T| \leq k - |A \cup V(e)| - |\{y\}| = k - (a + 1) - 1 = a - 1$, contradiction. $\square$

We now prove a condition that guarantees two k-contractible edges in any spanning tree.

Lemma 3 *Let Q be a spanning tree of a noncomplete graph G of connectivity k, set $\mathfrak{S} := \{V(e) : e \in E(Q)\}$, suppose that all $\mathfrak{S}$-fragments have cardinality at least $\frac{k-1}{2}$, set $\mathfrak{R} := \{V(e) : e \in E(Q), |V(e) \cap A| = 1$ for some $\mathfrak{S}$-end $A\}$. Then Q contains two k-contractible edges or there is an $\mathfrak{R}$-end of cardinality $\frac{k-1}{2}$ such that Q contains no k-contractible edge with an endvertex from it.*

Proof. Call a fragment C *small* if $|C| = \frac{k-1}{2}$ and *big* otherwise; call C *good* if Q contains a k-contractible edge that has at least one endvertex in C and *bad* otherwise. Call C *very good* if all edges from Q having exactly one endvertex in C are k-contractible. In this language, Theorem 8 tells us that an $\mathfrak{R}$-end B is small, or good, or $N_G(B)$ contains a small, very good $\mathfrak{R}$-fragment.

We may assume that Q contains at least one non-k-contractible edge, so that there exists an $\mathfrak{S}$-fragment and hence an $\mathfrak{S}$-end, which we call A. Take an $\mathfrak{S}$-end

$A' \subseteq \overline{A}$. There exist edges $e, e' \in E(Q)$ with $|V(e) \cap A| = 1$ and $|V(e') \cap A'| = 1$; clearly, $e \neq e'$, and we are done if both e, e' are k-contractible. In what remains we thus may assume that there *exists* an $\mathfrak{R}$-fragment.

Suppose that there exists a very good $\mathfrak{R}$-fragment C. Consider an $\mathfrak{R}$-end $B \subseteq \overline{C}$. If B is good, then there exists a k-contractible edge e having a vertex in common with B and another one having a vertex in common with C, which proves the lemma. Hence B is bad. If B is small, then the lemma is proved again, so we may assume that B is big. By Theorem 8, $N_G(B)$ contains a (small) very good $\mathfrak{R}$-fragment D. As C and D are disjoint and their union is not equal to $V(G)$, Q contains two distinct k-contractible edges that are incident with vertices from C or D, which proves the lemma.

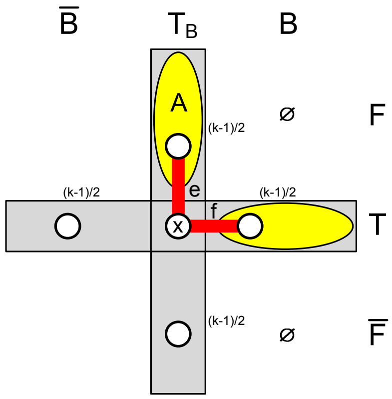
Therefore, we may assume that there are no very good $\mathfrak{R}$-fragments. Consequently, by Theorem 8, every big $\mathfrak{R}$-end is good. Moreover, we may assume that every small $\mathfrak{R}$-end is good, for otherwise we are done. Hence, *every* $\mathfrak{R}$-end B is good, and, for any $\mathfrak{R}$-end C contained in $\overline{B}$, Q contains two distinct k-contractible edges that have an endvertex in B and C, respectively, which gives the lemma. □.

We now specialize to DFS trees. A *DFS tree* of some graph G is a spanning tree Q with a prescribed *root vertex* r such that for every vertex x (including the case $x = r$), any two x-branches are nonadjacent in G, where an *x-branch* is the vertex set of any component of $Q - x$ that does not contain r. Now we are prepared to prove the following theorem, which implies Theorem 3 immediately, as, in a graph of connectivity k and of minimum degree at least $\frac{3}{2}k - \frac{3}{2}$, every fragment has cardinality at least $\frac{k-1}{2}$.

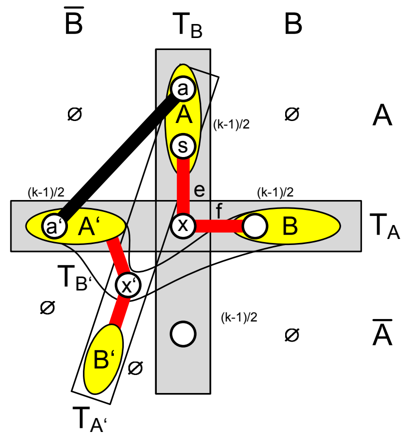
Theorem 9 *Let Q be a DFS tree of a noncomplete graph G of connectivity $k > 3$, and set $\mathfrak{S} := \{V(e) : e \in E(Q)\}$. (i) If all $\mathfrak{S}$-fragments have cardinality at least $\frac{k-1}{2}$, then Q contains at least one k-contractible edge. (ii) If all fragments have cardinality at least $\frac{k-1}{2}$, then Q contains at least two k-contractible edges.*

Proof. First, let us assume that all $\mathfrak{S}$-fragments have cardinality at least $\frac{k-1}{2}$. Set $\mathfrak{R} := \{V(e) : e \in E(Q), |V(e) \cap A| = 1$ for some $\mathfrak{S}$-end $A\}$ and observe that $\frac{k-1}{2} > 1$. By Lemma 3, we may assume that there exists a T_B-$\mathfrak{R}$-end B with $|B| = \frac{k-1}{2}$ such that Q contains no k-contractible edge with an endvertex from B. There exists an edge $e \in E(Q)$ and a T_A-$\mathfrak{S}$-end A such that $|V(e) \cap A| = 1$ and $V(e) \subseteq T_B$. By Claim 1 in the proof of Theorem 8, we see that $|A| = \frac{k-1}{2}$ and $A \subseteq T_B$. Let x be the vertex in $V(e) \setminus A$.

Observe that B is an $\mathfrak{S}$-end (as it is even an $\mathfrak{S}$-atom) and consider any edge $f \in E(Q)$ with $|V(f) \cap B| = 1$ (there exists at least one such edge). Let y be the vertex in $V(f) \setminus B$. As f is not k-contractible, there exists a $T \in \mathfrak{T}(G)$ with $V(f) \subseteq T$. According to Claim 1 in the proof of Theorem 8, $B \subseteq T$. All vertices from B are neighbors of A (for otherwise $(T_B - A) \cup (N_G(A) \cap B)$ would be a separating vertex set of G with less than k vertices), that is, $B \subseteq T_A$.



(a) Claim (i)



(b) Claim (ii)

Figure 3: The structure of the $\mathfrak{R}$-fragments B and F in Theorem 9.

Let F be a T-fragment. Then $F \cap \overline{B} \neq \emptyset$ implies $|F \cap T_B| \geq^\star |B \cap T| = \frac{k-1}{2}$ and $F \cap \overline{B} = \emptyset$ implies $|F \cap T_B| = |F| \geq \frac{k-1}{2}$, too. As the same holds for $\overline{F}$ instead of F, we deduce $|F \cap T_B| = |\overline{F} \cap T_B| = \frac{k-1}{2}$ and $T \cap T_B = \{y\}$. It follows that $A \subseteq F$ or $A \subseteq \overline{F}$, for otherwise T contains a vertex from A, as $G[A]$ is connected; this vertex can only be y, so that applying Claim 1 of the proof of Theorem 8 on f and A (or alternatively using Theorem 5 with $S = V(f)$) implies $A \subseteq T$, which gives the contradiction $|T \cap T_B| \geq |A| > 1$. It follows that $A = F \cap T_B$ or $A = \overline{F} \cap T_B$ and $x \in T \cap T_B$, so that $x = y$ (see Figure 3a).

Since $f \in E(Q)$ has been chosen arbitrarily from the edge neighborhood of B, we see that x is the only neighbor of B in Q. In particular, A is an $\mathfrak{R}$-end (certified by any f as above), since $B \cup \{x\} \subseteq T_A$. Let us take over the notion of small, big, good, bad, and very good fragments from Lemma 3; then A and B are small and B is bad. If A was bad, too, then by symmetry of A and B we see that $x = y$ is the only neighbor of A in Q. Since $G[A]$ and $G[B]$ are connected, each of them contains at most one x-branch of Q; if each of them contains an x-branch, these x-branches span $G[A]$ and $G[B]$ and, as every vertex in A has a neighbor in B and vice versa, the two x-branches are adjacent, which is a contradiction. Hence, one of $G[A], G[B]$ does not contain an x-branch, and, thus, contains the root of Q, whereas the other one is spanned by a single x-branch. By symmetry, let us assume that B contains the root, and consider any vertex $z \in \overline{B} \cap T_A$. Again, z and the vertices in A are in different x-branches, but z must have at least one neighbor in A, contradiction.

Therefore, we may assume from now on that A is good. Hence, Q contains a k-contractible edge aa' with $a \in A$ and $a' \in \overline{B} \cap T_A$ (see also Figures 1a and 1b). This proves (i) of the statement.

Now let us assume, in addition, that *all* fragments have cardinality at least $\frac{k-1}{2}$, and (reductio ad absurdum) that aa' is the unique k-contractible edge of Q. Observe that all small fragments of G are atoms, so that, by Theorem 6, they are either subsets of or disjoint from any smallest separating set.

Let us apply Theorem 8 to an arbitrary $\mathfrak{R}$-end B' such that $B' \subseteq \overline{A}$ (see Figure 3b). If B' was good, then Q would contain a k-contractible edge that has an endvertex in B', and is thus distinct from aa', contradiction. Hence B' is bad. If B' was big, then $N_G(B')$ would contain a small, very good $\mathfrak{R}$-fragment C by Theorem 8. Since aa' is the unique k-contractible edge in Q, we see that $a' \in C$, and, as C is an atom and $a' \in T_A$, we infer $C \subseteq T_A$ by Theorem 6. Since $V(e)$ is not k-contractible, $x \notin C$, which implies that C contains no vertex from B either. As $B \cup \{x\} \subseteq T_A$, it follows $C = T_A \cap \overline{B}$. Consequently, x is the only neighbor of $\overline{A}$ in Q, as $\overline{A}$ is bad, C is very good and x is the only neighbor of B in Q. Using these restrictions, it is now not possible to locate the root vertex of our DFS tree: It cannot be in $\overline{A} \cup \{x\}$, because then we find adjacent vertices from A and B in distinct x-branches, it cannot be in $A \cup C$, because then we find adjacent vertices from B and $\overline{A} \cap T_B$ in distinct x-branches, and it cannot be in B, because then we find adjacent vertices from C and $N_G(C) \cap \overline{A}$ in distinct x-branches — contradiction.

Therefore, B' is a small, bad $\mathfrak{R}$-end, just as B. We may infer — just as before for B — that $T_{B'} := N_G(B')$ contains a small, good $T_{A'}$-$\mathfrak{R}$-end A'. Since $A' \subseteq T_{B'} \subseteq T_A \cup \overline{A}$, we see that A and A' are disjoint. As A' is good and aa' is the only k-contractible edge, it follows $a' \in A'$, and, as A' is an atom having a vertex in common with T_A, we see that $A' \subseteq T_A$ by Theorem 6. Since A' contains $a' \notin T_B$, we infer for the same reason that A' does not contain x, and, as $G[A']$ is connected and $a' \in \overline{B}$, A' does not contain any vertex from B. It follows $A' = \overline{B} \cap T_A$. Moreover, we get, as for A and B above, a unique vertex x' in $T_{A'} \cap T_{B'}$ and infer that all non-k-contractible edges from Q with exactly one vertex from B' are incident with x'.

We claim that aa' is the only edge from Q that connects A and A'. Assume to the contrary there was another one, say, $zz' \in E(Q) \setminus \{aa'\}$, with $z \in A$ and $z' \in A'$. Since zz' is not k-contractible, there exists a $T' \in \mathfrak{T}(G)$ with $z, z' \in T'$, and from Theorem 6 we get $A \cup A' \subseteq T'$. Since $A \subseteq T'$, T' contains at least one vertex from $\overline{A}$ (by applying Lemma 2 to T' and T_A) and thus consists of the $k-1$ vertices from $A \cup A'$ and another vertex from $\overline{A}$; but then $T_A \setminus T' = B \cup \{x\}$ induces a connected subgraph of G, so T' does not separate T_A, contradiction.

Observe that (when the position of the root in Q is neglected), the situation is symmetric in A, B, a, x and A', B', a', x'. We have seen that $N_Q(B) = \{x\}$, $N_Q(A) = \{x, a'\}$, $N_Q(A') = \{x', a\}$, and $N_Q(B') = \{x'\}$. It follows that $N_Q(B \cup A \cup A' \cup B') = \{x, x'\}$. Let us again analyze the position of the root vertex r of our DFS tree. We claim that the following statement holds:

(*) $r \neq x$ and the r, x-path in Q enters x by an edge incident to A or B.

If (*) is false, $e \in E(Q)$ implies that the vertex s from $V(e) \setminus \{x\}$ and any vertex in $N_G(s) \cap B$ are in distinct adjacent x-branches (contradiction). Likewise, the following holds.

(*') $r \neq x'$ and the r, x'-path in Q enters x' by an edge incident to A' or B'.

Setting $X := B \cup A \cup A' \cup B'$ we thus see that $r \in X$, for otherwise the second last vertex of every r, X-path in Q (ending at the first occurrence of a vertex from X) is either x or x' and its r, x- or r, x'-subpath violates (*) or (*'), respectively. Furthermore, $x \neq x'$, for otherwise $r \in B$ violates (*'), $r \in A$ violates (*') if the r, x'-path in Q does not use the edge aa' and $r \in A$ violates (*) if it does, and we get symmetric violations for $r \in B'$ and $r \in A'$, respectively. Now we claim that $R := Q[X \cup \{x, x'\}]$ is connected, that is, a subtree of Q. Suppose, to the contrary, that R contains a vertex z such that there is no r, z-path in R. The r, z-path in Q therefore has to use vertices from $V(G) \setminus V(R)$, and, therefore, both x and x'. If x is used last (that is, x' is on the r, x-path), then (*) is violated, and otherwise (*') is.

By symmetry of A, B and A', B', we may assume that r is not in B. Since $|T_B \cap \overline{A}| = \frac{k-1}{2} > 1$, there exists a vertex $t \in (T_B \cap \overline{A}) \setminus \{x'\}$; t has a neighbor $z \in B$, and t, z cannot be in different v-branches for any vertex v, so that either z is on the r, t-path in Q or t is on the r, z-path in Q. The first option cannot occur, as B has only one neighbor in Q, and the second one implies that t, like *all* vertices from the r, z-path, is in $V(R)$. Since $t \notin \{x, x'\}$ and $t \notin B \cup A \cup A'$, we deduce $t \in B'$, in other words, $T_B \cap B' \neq \emptyset$. By Theorem 6, $B' \subseteq T_B$, which determines $T_B = A \cup \{x\} \cup B'$, $T_A = A' \cup \{x\} \cup B$, $T_{A'} = A \cup \{x'\} \cup B'$, and $T_{B'} = A' \cup \{x'\} \cup B$. The set $Y := \overline{A} \cap \overline{B} \cap \overline{A'} \cap \overline{B'}$ has all its neighbors in $T_A \cup T_B \cup T_{A'} \cup T_{B'} \setminus (A \cup B \cup A' \cup B') = \{x, x'\}$, which implies that Y is empty. Therefore, $\overline{A} \cap \overline{B} = \{x'\}$, so that $N_G(x') = A' \cup B' \cup \{x\}$. Hence $\{x'\}$ is a fragment of cardinality 1, contradiction. □

4 Contractible edges in DFS trees of 3-connected graphs

For proving Theorem 1, observe that the fragment size conditions from Theorem 9 are vacuously true in the case that $k = 3$. However, the conclusion in (ii) does not hold for that case, as shown for its direct implication Theorem 3 in the introduction, so we may expect some differences in the argumentation for $k = 3$. Nonetheless, a substantial part of the proof of Theorem 8 can be taken over.

Proof of Theorem 1. Let Q be a DFS tree of a 3-connected graph G nonisomorphic to K_4. We will prove that Q contains at least one 3-contractible edge, and that Q contains at least two 3-contractible edges unless G is a prism or a prism plus a single edge (and Q has a special shape). Set $\mathfrak{S} := \{V(e) : e \in$

$E(Q)\}$ and $\mathfrak{R} := \{V(e) : e \in E(Q), |V(e) \cap A| = 1$ for some $\mathfrak{S}$-end $A\}$. Let us adopt once more the terminology of small, big, good, bad, and very good fragments introduced in the proof of Lemma 3.

By Lemma 3 (which is, in contrast to Theorem 9, true for $k = 3$, too), we may assume that there exists a small, bad T_B-$\mathfrak{R}$-end $B = \{b\}$. There exists an edge $e \in E(Q)$ and a T_A-end A such that $V(e) \subseteq T_B$ and $|V(e) \cap A| = 1$, and as in Claim 1 in the proof of Theorem 8 we see that $|A| = 1$, say, $A = \{a\}$. Let x be the vertex in $V(e) \setminus \{a\}$ and let t be the vertex in $T_B \setminus V(e)$. As bt is 3-contractible (as any smallest separating set containing b and t would have both neighbors a and x of b on the same side, since $\{a, b, x\}$ induces a triangle in $G \not\cong K_4$), we know that $bt \notin E(Q)$, as otherwise B would not be bad. Hence, as $ax \in E(Q)$, exactly one of ba, bx is in $E(Q)$. Observe that a is not adjacent to t, because a has degree 3 and a neighbor in $\overline{B}$.

Suppose that $ba \in E(Q)$. Then there exists a $T \in \mathfrak{T}(G)$ containing b, a, and a vertex p from $\overline{B}$. Consider any T-fragment F. At least one of $F \cap \overline{B}, \overline{F} \cap \overline{B}$ must be empty, for otherwise one of these sets is an $\{a, p, x\}$-fragment and the other one is an $\{a, p, t\}$-fragment; this is not possible, because a has only one neighbor in $\overline{B}$. So $\{x\}$ or $\{t\}$ is a T-fragment, but the latter is not because t is not adjacent to $a \in T$. Consequently, $\{x\}$ is a T-fragment, and, in fact, an $\mathfrak{S}$-atom.

Therefore, after possibly having exchanged the names of a, x (and resetting A to $\{a\}$ accordingly), we may assume without loss of generality that $bx \in E(Q)$; then T_A consists of b, x, and a vertex s from $\overline{B}$ (replacing the former vertex p), and we see immediately that $A = \{a\}$ is a small $\mathfrak{R}$-end, too. The edge as is 3-contractible, because $N_G(a) \setminus \{s\} = \{b, x\}$ induces a complete graph in G.

Now if the small $\mathfrak{R}$-end A was bad, ax would be the only edge incident with a in Q, since as is 3-contractible. Then one of a and b must be the root vertex r of the DFS tree Q, because otherwise a, b would belong to different x-branches but are adjacent. If $r = a$, then b and t belong to different x-branches but are adjacent; if otherwise $r = b$, then a and s belong to different x-branches but are adjacent. Thus, A is good.

Since $ab \notin E(Q)$ and ax is not 3-contractible, it follows that Q contains the 3-contractible edge as. This gives the first claim; for the second, let us assume that as is the only 3-contractible edge from Q. We have to prove that G is either the prism or the prism plus a single new edge; to this end, we proceed as in the proof of Theorem 9, and consider an arbitrary $T_{B'}$-$\mathfrak{R}$-end $B' \subseteq \overline{A}$. If it was good, we would find a 3-contractible edge having an endvertex in common with B', and, thus, distinct from as, which gives a contradiction. Thus, B' is bad. If B' was big, it would contain a very good end in its neighborhood; just as in the proof of Theorem 9, we cannot locate the root of Q properly, which gives a contradiction.

Therefore, B' is a bad, small $\mathfrak{R}$-end, and we may apply to B' the same line of arguments that we applied before to B: $T_{B'}$ consists of three vertices a', x', t' (we will use these letters also for any other arbitrarily chosen $\mathfrak{R}$-end later), where $A' := \{a'\}$ is a good, small $T_{A'}$-$\mathfrak{R}$-end, and $T_{A'}$ consists of b', x' and a vertex s' from $\overline{B'}$, where, moreover, $x'a'$ and $x'b'$ are from $E(Q)$ and not 3-contractible, and $a's'$ is from $E(Q)$ and 3-contractible.

Since as is the only 3-contractible edge, we see that $a's' = as$, which implies $a' = s$ and $s' = a$. Again the situation is symmetric in a, b, x, s, t and a', b', x', s', t', and we may proceed almost literally as in the proof of Theorem 9 by showing first that $x \neq x'$, $r \notin \{x, x'\}$, and $R := Q[\{a, b, x, a', b', x'\}]$ is a subtree of Q, and thus, more precisely, a path $bxaa'x'b'$. By symmetry, we may assume that $r \in \{a', b'\}$, and, as b cannot be on the r,t-path in Q, that t must be on the r,b-path in Q and hence in $R \cap \overline{A}$. Different from the more general argument, we now have two options for t: t is either b', or it is x'. We consider the corresponding cases separately:

Case 1. $t = b'$

Then $t' = b$. The set $X := \overline{A} \cap \overline{B} \cap \overline{A'} \cap \overline{B'}$ has all its neighbors in $T_A \cup T_B \cup T_{A'} \cup T_{B'} \setminus (A \cup B \cup A' \cup B') = \{x, x'\}$. Therefore, X is empty, and so $\overline{A} \cap \overline{B} = \{x'\}$, which implies that G is the prism.

Case 2. $t = x'$.

Assume that $a, b, a', b', x, x' = t, t'$ are the only vertices of G and that they are all distinct. Since $T_{A'} = \{b', x', a\}$ is a smallest separating set and $t' \notin T_{A'}$, $st' \notin G$. Since t' has degree at least three and is adjacent to b', we conclude $N_G(t') = \{x, t, b'\}$. As certified by the $\mathfrak{S}$-end $\{b'\}$ and the edge $b't$, $\{t'\}$ is an $\mathfrak{R}$-end contained in $\overline{A}$. As we have seen above (for the arbitrarily chosen B'), the neighborhood of such a fragment necessarily contains a', contradiction.

If there are further vertices other than those listed in the previous paragraph, they are all from $X := \overline{A} \cap \overline{B} \cap \overline{A'} \cap \overline{B'}$. Then X has all its neighbors in $T_A \cup T_B \cup T_{A'} \cup T_{B'} \setminus (A \cup B \cup A' \cup B') = \{x, x', t'\} =: T$. Therefore, $t' \neq x$, and X is, indeed, an T-fragment, and $\overline{X} = \{a, b, a', b'\}$. Since t' has only one neighbor in $\overline{X}$, $F := X \cup \{t'\}$ is an $\{x, x', b'\}$-fragment (with $\overline{F} = \{a, b, a'\}$). The $\mathfrak{S}$-end $\{b'\}$ together with the edge $b't$ certifies that F is an $\mathfrak{R}$-fragment, and, thus, contains an $\mathfrak{R}$-end B'', which is in $\overline{A}$. However, the neighborhood of B'' does not contain a', as it should, by what we have proven about the arbitrarily chosen B' above, which gives a contradiction.

Therefore, there are no further vertices but $a, b, a', b', x, x' = t, t'$, and the vertices listed are not all distinct. This implies $x = t'$. The neighborhoods of a, b, a', b' are determined, so that $E(G)$ is determined up to a possible edge connecting x and x'. If x and x' are not adjacent, we get the prism, and otherwise we get the prism plus a single edge.

This proves the theorem. □

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