Dynamics of Cycles in Polyhedra I: The Isolation Lemma

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Abstract

A cycle C of a graph G is *isolating* if every component of $G - V(C)$ consists of a single vertex. We show that isolating cycles in polyhedral graphs can be extended to larger ones: every isolating cycle C of length $6 \leq |E(C)| < \left\lfloor \frac{2}{3}(|V(G)| + 4) \right\rfloor$ implies an isolating cycle C' of larger length that contains $V(C)$. By "hopping" iteratively to such larger cycles, we obtain a powerful and very general inductive motor for proving long cycles and computing them (we will give an algorithm with quadratic running time). This is the first step towards the so far elusive quest of finding a universal induction that captures longest cycles of polyhedral graph classes.

Our motor provides also a method to prove linear lower bounds on the length of Tutte cycles, as C' will be a Tutte cycle of G if C is. We prove in addition that $|E(C')| \leq |E(C)| + 3$ if G contains no face of size five, which gives a new tool for results about cycle spectra, and provides evidence that faces of size five may obstruct many different cycle lengths. As a sample application, we test our motor on the following so far unsettled conjecture about essentially 4-connected graphs.

A planar graph is *essentially* 4-*connected* if it is 3-connected and every of its 3-separators is the neighborhood of a single vertex. Essentially 4-connected graphs have been thoroughly investigated throughout literature as the subject of Hamiltonicity studies. Jackson and Wormald proved that every essentially 4-connected planar graph G on n vertices contains a cycle of length at least $\frac{2}{5}(n+2)$, and this result has recently been improved multiple times, culminating in the lower bound $\frac{5}{8}(n+2)$. However, the currently best known upper bound is given by an infinite family of such graphs in which no graph G contains a cycle that is longer than $\left\lfloor \frac{2}{3}(n+4) \right\rfloor$; this upper bound is still unmatched.

Using isolating cycles, we improve the lower bound to match the upper. This settles the long-standing open problem of determining the circumference of essentially 4-connected planar graphs. All our results are tight.

keywords: Isolating Long Cycles, Circumference, Tutte Cycle Dynamics, Polyhedral Graphs, 3-Connected Planar Graphs, Essentially 4-Connected, Discharging

*This research is supported by the grant SCHM 3186/2-1 (401348462) from the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation), and within the project 57447800 by DAAD (as part of BMBF, Germany) and the Ministry of Education Science, Research and Sport of the Slovak Republic.

1 Introduction

One of the unchallenged milestones in graph theory is the result by Tutte [14] in 1956 that every 4-connected planar graph G is Hamiltonian, i.e. has circumference $|V(G)|$, where the *circumference* $circ(G)$ of a graph G is the length of a longest cycle of G. However, decreasing the connectedness assumption from 4 to 3 reveals infinitely many planar graphs that do not have long cycles: In fact, Moon and Moser [12] showed that there are infinitely many 3-connected planar (i.e. *polyhedral*) graphs G that have circumference at most $9n^{\log_3 2}$ for $n := |V(G)|$, and this upper bound is best possible up to constant factors, as there is a constant $c > 0$ such that every polyhedral graph contains a cycle of length at least $cn^{\log_3 2}$ [3].

One of the biggest remaining open problems in this area ever since is to characterize properties *between* connectivity 3 and 4 that imply long cycles (see Grünbaum and Walther [10, Theorem 1] for a classic reference that summarizes an abundance of such results for polyhedral subclasses). Essential 4-connectedness is such a property and will be a focus of this paper.

Indeed, essentially 4-connected graphs have been thoroughly investigated throughout literature for this purpose. Other traditional purposes include reduction techniques for important graph-theoretic problems like the four-color problem (as, for 3-regular graphs on at least seven vertices, essentially 4-connectivity is equivalent to cyclically 4-edge-connectivity). An upper bound for the circumference of essentially 4-connected planar graphs was given in [4] by an infinite family of such graphs on $n \geq 14$ vertices in which every graph G satisfies $circ(G) = \lfloor \frac{2}{3}(n+4) \rfloor$; the graphs in this family are in addition maximal planar. Regarding lower bounds, Jackson and Wormald [11] proved in 1992 that $circ(G) \geq \frac{2}{5}(n+2)$ for every essentially 4-connected planar graph G on n vertices. Fabrici, Harant and Jendroľ [4] improved this lower bound to $circ(G) \geq \frac{1}{2}(n+4)$; this result in turn was recently strengthened to $circ(G) \geq \frac{3}{5}(n+2)$ [5], and then further to $circ(G) \geq \frac{5}{8}(n+2)$ [6]. For the restricted case of maximal planar essentially 4-connected graphs G, the matching lower bound $circ(G) \geq \frac{2}{3}(n+4)$ was proven very recently in [7]; however, the methods used there are specific to maximal planar graphs. For the general polyhedral case, it is still an open conjecture that every essentially 4-connected planar graph G on n vertices satisfies $circ(G) \geq \lfloor \frac{2}{3}(n+4) \rfloor$; while this conjecture has been an active research topic at workshops (such as the ILKE Workshops on Graph Theory) for over a decade[1], it was only recently explicitly stated in [7, Conjecture 2].

Here, we show that $circ(G) \geq \lfloor \frac{2}{3}(n+4) \rfloor$ for every essentially 4-connected planar graph G. This matches the upper bound given above tightly. In fact, we give a much more general result, of which the previous statement is just an implication: Let a cycle C of a graph G be *isolating*[2] if every component of $G - V(C)$ consists of one (isolated) vertex. Instead of regarding essentially 4-connected polyhedral graphs, the result holds for all polyhedral graphs that contain an isolating cycle, and instead of just proving high circumference, we prove many different large cycle lengths as follows. Let a cycle C of a polyhedral graph G be *extendable* if G contains a larger isolating cycle C' such that $V(C) \subset V(C')$ and

[1] personal communication with Jochen Harant

[2] Such cycles are sometimes also called *dominating*, but this term is used inconsistently throughout literature.

$|V(C')| \leq |V(C)| + 3 + n_5(G)$, where $n_5(G)$ is the number of faces of size five in any planar embedding of G. By a result of Whitney [15], $n_5(G)$ depends only on G and not on the planar embedding. The following is our main result.

Lemma 1 (Isolation Lemma)**.** *Every isolating cycle of length $c < \min\{\lfloor \frac{2}{3}(n+4) \rfloor, n\}$ in a polyhedral graph G on n vertices is extendable.*

We note that the assumption $c < n$ may equivalently be replaced with $n \geq 6$, as we have $\lfloor \frac{2}{3}(n+4) \rfloor \leq n$ if and only if $n \geq 6$. While one part of the proof scheme for Lemma 1 follows the established approach of using Tutte cycles in combination with the discharging method, we contribute an intricate intersection argument on the weight distribution between sets of neighboring faces, which we call tunnels. This method differs substantially from the ones used in [4, 7, 5, 6] (for example, [7] exploits the inherent structure of maximal planar graphs) and is able to harness the dynamics of extending cycles in general polyhedral graphs. In particular, we will discharge weights along an unbounded number of faces, which was an obstacle that was not needed to overcome for the previous bounds.

Although the Isolation Lemma may look like a distant polyhedral relative of Woodall's Hopping Lemma [17], these lemmas are not very similar: while the Hopping Lemma only allows cycle extensions when there is a common neighbor of two cycle vertices having distance two, the Isolation Lemma allows a variety of much more general extensions, but is inherently limited to planarity. In fact, the Isolation Lemma fails hard for non-planar graphs, as the graphs $K_{c,n-c}$ for any $c \geq 3$ and $n > 2c$ and any (isolating) cycle of length $2c$ in these show. For $c = 2$, these graphs are planar and show that the Isolation Lemma requires that G is 3-connected.

We state some immediate corollaries of the Isolation Lemma.

Corollary 2. *Let G be a polyhedral graph on $n \geq 6$ vertices with n_5 faces of size five. If G contains an isolating cycle C, G contains isolating cycles of at least $(\lfloor \frac{2}{3}(n+4) \rfloor - |E(C)| + 1)/(3 + n_5)$ different lengths in $\{|E(C)|, \ldots, \lfloor \frac{2}{3}(n+4) \rfloor\}$, all of which contain $V(C)$.*

In particular, for bipartite graphs, Corollary 2 implies the following.

Corollary 3. *If a bipartite polyhedral graph contains an isolating cycle C, it contains an isolating cycle of every even length $l \in \{|E(C)|, \ldots, \lfloor \frac{2}{3}(n+4) \rfloor\}$.*

In other words, bipartite polyhedral graphs with an isolating cycle are bipancyclic in the given range. In view of the sheer number of results in Hamiltonicity studies that use subgraphs involving faces of size five (consider for instance the Tutte Fragment or the fragment of Faulkner and Younger for classic examples), Lemma 1 provides evidence for why these faces are indeed important: they are key to having a small number of different cycle lengths if a small isolating cycle exists. Another corollary of Lemma 1 is that polyhedral graphs on n vertices, in which all cycles have length less than $\min\{\lfloor \frac{2}{3}(n+4) \rfloor, n\}$, do not contain any isolating cycle; for example, this holds for the sufficiently large Moon-Moser graphs [12] and for every of the 18 graph classes of [10, Theorem 1] that have shortness exponent less than 1.

Finally, the Isolation Lemma implies also the following corollary.

Corollary 4. *Every essentially 4-connected planar graph G on n vertices contains an isolating Tutte cycle of length at least* $\min\{\lfloor \frac{2}{3}(n+4) \rfloor, n\}$.

Proof. It is well-known that every 3-connected planar graph on at most 10 vertices is Hamiltonian [1]; these graphs contain in particular the essentially 4-connected planar graphs. Since every Hamiltonian cycle is isolating, this implies the theorem for every $n \leq 10$. We therefore assume $n \geq 11$; in particular, $\lfloor \frac{2}{3}(n+4) \rfloor \leq n$. For $n \geq 11$, it was shown in [4, Lemma 4(i)] that G contains an isolating Tutte cycle C. Applying iteratively the Isolation Lemma to C gives the claim and preserves a Tutte cycle, as no vertex of C is deleted. □

Corollary 4 encompasses and strengthens most of the results known for the circumference of essentially 4-connected planar graphs, some of which can be found in [4, 9, 18]. At the same time and of independent interest, Corollary 4 allows to extend every isolating Tutte cycle C of a polyhedral graph G to an isolating Tutte cycle of G of linear length in $|V(G)|$. We would like to address that, during the review phase of this paper, Wigal and Yu proposed an independent proof of Corollary 4 using different methods [16].

We give polynomial-time algorithms that compute cycles of the lengths claimed by Lemma 1 and Corollary 4 at the end of this paper. The efficiency of these algorithms are in stark contrast to the well-known problem of computing the circumference itself, which is NP-hard even for 3-regular polyhedral graphs [8]. As a relaxation of the classic longest cycle problem, we conjecture that for many[3] planar graph classes $\mathcal{G}$, a cycle of length at least $circ(\mathcal{G}) := \inf_{G \in \mathcal{G}} \frac{circ(G)}{|V(G)|}$ can be computed in polynomial time for every graph $G \in \mathcal{G}$.

For the class of essentially 4-connected planar graphs, we show that this can be done in time $O(|V(G)|^2)$.

2 Preliminaries

We use standard graph-theoretic terminology and consider only graphs that are finite, simple and undirected. For a vertex v of a graph G, denote by $\deg_G(v)$ the degree of v in G. We omit subscripts if the graph G is clear from the context. Two edges e and f are *adjacent* if they share one end vertex. We denote a (not necessarily simple) path of G that visits the vertices $v_1, v_2, \ldots, v_i$ in the given order by $v_1 v_2 \ldots v_i$.

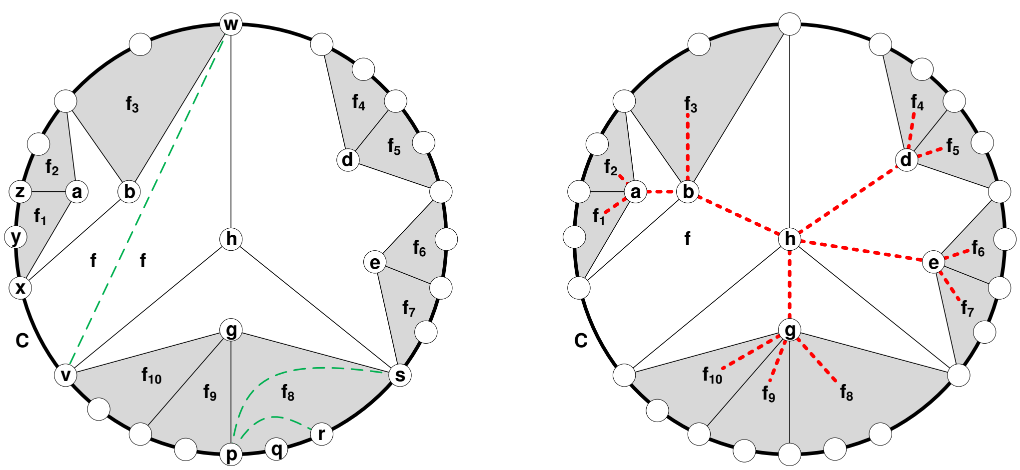
A *separator* S of a graph G is a subset of $V(G)$ such that $G - S$ is disconnected; we call S a *k-separator* if $|S| = k$. Let a cycle C of a graph G be *isolating* if every component of $G - V(C)$ consists of a single vertex (see Figure 1a for an example). We do not require that these single vertices have degree three (this differs for example from [4, 5, 6]). A *chord* of a cycle C is an edge $vw \notin E(C)$ for which v and w are in C. By a result of Whitney [15], every 3-connected planar graph has a unique embedding into the plane (up to flipping and the choice of the outer face). Hence, we assume in the following that such graphs are equipped with a fixed planar embedding, i.e. are *plane*, so that vertices are points and edges are point sets. Let $F(G)$ be the set of faces of a plane graph G.

[3] but not all, as computing Hamiltonian cycles in Hamiltonian graphs is still NP-hard

3 Proof of the Isolation Lemma

Let $G = (V, E)$ be a 3-connected plane graph on n vertices, and let C be an isolating cycle of G of length $c < \min\{\lfloor \frac{2}{3}(n+4) \rfloor, n\}$. We assume to the contrary that C is not extendable. Then $c < \frac{2}{3}(n+3)$, as $c \geq \frac{2}{3}(n+3)$ implies $c \geq \lceil \frac{2}{3}(n+3) \rceil = \lfloor \frac{2}{3}(n+4) \rfloor$ for the integers c and n, which is a contradiction.

Let V^- be the subset of V that is contained in one of the two *regions* (i.e. maximal path-connected open sets) of $\mathbb{R}^2 - C$, and let $V^+ := V - V(C) - V^-$. Without loss of generality, we assume $|V^-| \leq |V^+|$. Since $c < n$ and $|V^-| \leq |V^+|$, we have $V^+ \neq \emptyset$. Let H be the plane graph obtained from G by either deleting all chords of C if $V^- \neq \emptyset$ or otherwise deleting all chords of C whose interior point sets are contained in the same region of $\mathbb{R}^2 - C$ as V^+ (see Figure 1). Let $H^- := H - V^+$ and $H^+ := H - V^- - (E(H^-) - E(C))$. In particular, H^+ does not contain any chord of C (H^- may however), and both H^- and H^+ have a face with boundary C.



(a) An isolating cycle C (fat edges) of an essentially 4-connected plane graph G; vertices in V^+ are not drawn. Here, $V^- = \{a, b, d, e, g, h\}$ and H^- has no minor 1-face but would have one after contracting yz. The dashed chord vw of C is in G but not in H, so that f is a face of H (in fact, a thick major 1-face) but not a face of G. The minor face f_8 has exactly the two arches pgs and pr; note that ps is not an arch of f_8.

(b) The subgraph H^- of G (solid edges) and a tree T^- constructed from H^- (dashed edges). There are $|M^-| = 10 \geq |V^-| + 2 = 8$ minor faces in H^- (depicted in gray), each of which is a thick 2- or 3-face that corresponds to a leaf of T^-.

Figure 1

For a face f of G, H, H^- or H^+, the edges of C that are incident with f are called *C-edges* of f and their number is denoted by m_f. Such a face f is called *j-face* if $m_f = j$. A *C-vertex* of f is a vertex that is incident to a C-edge of f. A C-edge of f is *extremal* if it is adjacent to at most one C-edge of f, and *non-extremal* otherwise; a C-vertex of f is *extremal* if it is incident to at most one C-edge of f, and *non-extremal* otherwise. Let two faces f and g of H be *opposite* if f and g have a common C-edge. If f has a C-edge e, let the *e-opposite face* of f be the face of H that is different from f and incident to e.

Let a face f of H be *thin* if $f \in F(H^-)$ and $V^- = \emptyset$, and *thick* otherwise. A face f of H is called *minor* if it is either thin and incident to exactly one edge that is not in C or thick and incident to exactly one vertex that is not in C; otherwise, f is called *major* (this definition differs from [4, 5, 6]). Our interest in minor faces stems from the fact that all their C-vertices are consecutive in C, which we will later use to impose structural restrictions on the incident edges to these vertices. Let M^- and M^+ be the sets of minor faces of H^- and H^+, respectively. For a minor face f that has an odd number of C-vertices or C-edges, we call its unique C-vertex or C-edge in the middle the *middle* C-vertex or C-edge of f. For a minor thick face f of H, let v_f be the unique vertex of $V^+ \cup V^-$ that is incident to f.

We now show that the minor faces of H are precisely the leaf faces of two suitably defined trees T^- and T^+ that we construct from H^- and H^+, respectively. If $V^- = \emptyset$, let T^- be the weak dual of H^-. If $V^- \neq \emptyset$, let T^- (we define T^+ analogously from H^+) be a graph with vertex set $M^- \cup V^-$ and the following edge-set (see Figure 1b). First, for every face $f \in M^-$, add the edge fv_f to T^-. Second, for every major face f of H^- (in arbitrary order), fix any vertex $v \in V^-$ that is incident to f and add the edge vw to T^- for every vertex $w \in V^- - \{v\}$ that is incident to f. We prove that T^- and T^+ are trees with the following properties.

Lemma 5. *T^- and T^+ are trees on at least three vertices with leaf sets M^- and M^+, respectively. T^+ contains no vertex of degree two and, if $V^- \neq \emptyset$, the same holds for T^-.*

Proof. First, assume that $V^- \neq \emptyset$ (the proof for T^+ is analogous). Then H^- does not contain any chord of C. Consider any two inner faces f and g of H^- that are incident to a common edge e. Since e is no chord of C and C is isolating, f and g are incident to exactly one common vertex $v \in V^-$. By construction of T^-, all vertices of V^- that are incident to f or g are connected in T^-; in particular, $\deg_{T^-}(v) \geq \deg_G(v) \geq 3$. Hence, T^- is connected. As C is isolating, every two faces of H^- are incident to at most one common vertex of V^-. Hence, the union of the acyclic graphs that are constructed for every major face of H^-, and thus T^- itself, is acyclic. We conclude that T^- is a tree with inner vertex set V^-, leaf set M^- and no vertex of degree two.

Assume that $V^- = \emptyset$. Then H^- is connected and outerplanar, and it is well-known that the weak dual of such a graph is a tree. By planar duality, M^- is the set of vertices of degree one in T^-. It remains to show that $|V(T^-)| \geq 3$. By the result of the previous paragraph for T^+, T^+ contains at least three vertices and no vertex of degree two, so that $|M^+| \geq 3$. Consider any minor face $f \in M^+$ and its two extremal C-vertices v and w. Since G is polyhedral, $\{v, w\}$ is not a 2-separator of G, so that a non-extremal C-vertex of f is incident to a chord of C in H^-. Since $|M^+| \geq 3$, H^- contains at least two chords of C, which gives $|V(T^-)| \geq 3$. In particular, no face of H has boundary C. □

Note that T^- may contain vertices of unbounded degree even if every vertex of V^- has degree three in G (for example, $\deg_{T^-}(h) = 4$ in Figure 1b even if $\deg_G(g)$ would be three). We now relate the number of vertices in $V - V(C)$ to the number of minor faces of H.

Lemma 6. *$|M^-| \geq |V^-| + 2$ and $|M^+| \geq |V^+| + 2$.*

Proof. We only prove the first claim $|M^-| \geq |V^-| + 2$, as the second has an analogous proof. By Lemma 5, $|V(T^-)| \geq 2$. Every tree T on at least two vertices has exactly $2 + \sum_{v \in V(T),\ \deg(v) \geq 3} (\deg(v) - 2)$ leaves. If $V^- = \emptyset$, this implies the claim directly. If $V^- \neq \emptyset$, the claim follows from the formula above, as T^- has no vertex of degree two by Lemma 5. □

By definition of minor faces, H has no minor 0-face. Consider a minor 1-face f of H with C-edge vw; since G is simple, f is thick. Then the cycle obtained from C by replacing vw with the simple path vv_fw shows that C is extendable, which contradicts our assumption. We conclude that H has no minor 1-face. This implies $c \geq 6$, since $V^+ \neq \emptyset$ implies $|M^+| \geq 3$ by Lemma 6.

To summarize our assumptions, we know that C is not extendable, $6 \leq c < \min\{\frac{2}{3}(n + 3), n\}$, $|V^-| \leq |V^+|$, $|V^+| \geq 1$ and H has no minor 1-face. For the final contradiction to these assumptions, we aim to prove

$$2c \geq 4(|M^-| + |M^+|) \text{ if } V^- \neq \emptyset \text{ and} \tag{1}$$
$$2c \geq 2|M^-| + 4|M^+| \text{ if } V^- = \emptyset. \tag{2}$$

This contradicts our assumption $c < \frac{2}{3}(n + 3)$ by the following lemma.

Lemma 7. *Inequality (1) implies $c \geq \frac{2}{3}(n + 4)$ and Inequality (2) implies $c \geq \frac{2}{3}(n + 3)$.*

Proof. By Lemma 6, $|M^-| \geq |V^-| + 2$ and $|M^+| \geq |V^+| + 2$. Moreover, we have $|V^-| + |V^+| = n - c$. Hence, if $V^- \neq \emptyset$, Inequality 1 implies $c \geq 2(n - c + 4)$ and thus $c \geq \frac{2}{3}(n + 4)$. If $V^- = \emptyset$, we have $|V^+| = n - c$, so that Inequality 2 implies $c \geq 2(n - c + 3)$ and thus $c \geq \frac{2}{3}(n + 3)$. □

In the case $V^- \neq \emptyset$, Lemma 7 slightly strengthens the bound $\lfloor \frac{2}{3}(n + 4) \rfloor$ of both the Isolation Lemma and Corollary 4 to $\frac{2}{3}(n + 4)$. We note that the case $V^- = \emptyset$ played a special role in the discharging proofs of all of the previous results and had most often to be handled separately. Here, we build a common framework that integrates the arguments for this special case into the general one, but still distinguishes thin from thick faces.

In order to prove Inequality (1) or (2), we will charge every j-face of H with weight j; hence, the total charge has weight $2c$. Then we discharge (i.e. move) these weights to minor faces such that no face has negative weight. We will prove that after the discharging every minor face of H has sufficiently large weight (at least the coefficient given in the respective inequality) to satisfy Inequality (1) or (2). The only problem are minor 2- and 3-faces, as these are charged with weight less than 4. We will transfer sufficient weight to them, so that the problem shifts to large minor faces, for which we then examine their (local and non-local) neighborhood in order to find that C is extendable.

3.1 Arches and Tunnels

For a face f of H, a path A of G is an *arch* of f if f is minor and A is either

- a chord of C whose inner point set is strictly contained in f and that does not join the two extremal C-vertices of f (see *pr* in Figure 1a) or

- the simple path of maximal length in $H - E(C)$ all of whose edges are incident to f (see *pgs* in Figure 1a).

In the last case, we say that A is *proper*; then A has length one if f is thin and length two if f is thick. Hence, an arch A is proper if and only if $A \subseteq H$; in addition, every minor face f has exactly one proper arch. Since $|V(T^-)| \geq 3$ by Lemma 5, no two leaves of T^- are adjacent in T^-. Hence, every arch A is an arch of exactly one face of H, which we call the *face* $f(A)$ of A.

Let the *archway* of an arch A be the simple path in C between the two end vertices of A in which all edges are incident to $f(A)$. Since any arch A and its archway bound a face g in the graph $A \cup C$, we define $m_A := m_g$, A as a *j-arch* if $m_A = j$, A as *thick* if $f(A)$ is thick, and the (extremal) *C-vertices* and *C-edges* of A as the (extremal) C-vertices and C-edges of g. Since G is simple, every arch has exactly two extremal C-edges. By the last condition of the definition of arches, no two arches of f have the same archway (in fact, the archways of the arches of a face form a laminar family on $E(C)$). If f and g are arches or faces of H, let $m_{f,g}$ be the number of C-edges that f and g have in common; we say that f and g are *opposite* if $m_{f,g} > 0$. An *arch A of an arch B* is an arch of $f(B)$ such that every C-edge of A is a C-edge of B; we also say that B *has arch* A.

Consider the 3-arches $T_1, \dots, T_k$ in Figure 2 and assume for this paragraph that every T_i is thick and proper, so that every $f(T_i)$ is a minor 3-face. Since every $f(T_i)$ receives only initial weight 3 and k is unbounded, every local method of transferring weights to reach weight at least 4 per minor face is bound to fail. Unfortunately, Figure 2 is not the only example where non-local methods are needed: in fact, there are infinitely many configurations in which weights must be transferred non-locally. We will therefore design the upcoming discharging rule in such a way that weight transfers do not depend on faces but instead on arches; this will reduce all structures that have to be handled non-locally to one common non-local structure (called tunnel), which is essentially the one shown in Figure 2.

Let two 3-arches A and B be *consecutive* if $m_{A,B} = 1$. The reflexive and transitive closure of this symmetric relation partitions the set of all 3-arches whose middle C-edge is not a C-edge of a minor thin 2-face; we call the sets of this partition *tunnels* (see Figure 2). Since G is plane, G imposes a notion of clockwise and counterclockwise on C; in the following, both directions always refer to C. The *counterclockwise track* of a tunnel T (which will transfer weights counterclockwise around C) is the sequence $(T_1, T_2, \dots, T_k)$ of the 3-arches of T such that T_{i+1} is the clockwise consecutive successor of T_i for every $1 \leq i < k$.

The *exit pair* (g, e) of a counterclockwise track consists of the (in C) counterclockwise extremal C-edge e of T_1 and the e-opposite face g of $f(T_1)$. *Clockwise tracks* and their *exit pairs* are defined analogously. A *track* is a counterclockwise or clockwise track. We call both a track $(T_1, T_2, \dots, T_k)$ and its tunnel T *cyclic* if $k \geq 3$ and T_k and T_1 are consecutive, and *acyclic* otherwise. The *exit pairs* (g, e) and (g', e') of a tunnel T are the in total two exit pairs of the counterclockwise and clockwise tracks of T; we call g and g' *exit faces* of T. We have $e = e'$ if and only if T is cyclic, since $e = e'$ implies $k \neq 2$ due to $c \geq 6$; moreover, if $e = e'$, g' and g are opposite faces, so that $g \neq g'$. Hence, the exit pairs of an acyclic tunnel are always different, while its exit faces may be identical.

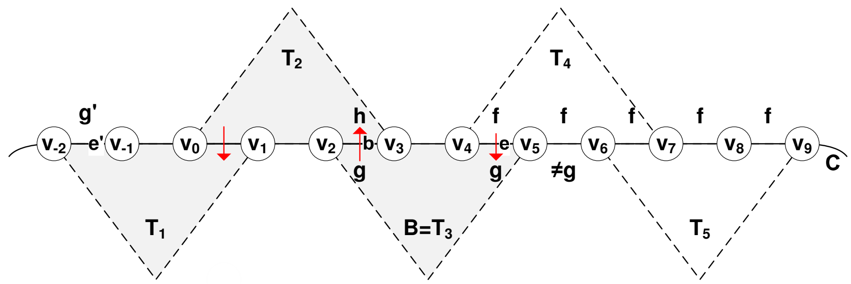


Figure 2: An acyclic counterclockwise track $T = (T_1, T_2, T_3, T_4, T_5)$ of a tunnel and the exit pair (g', e') of T. Here, (g', e') is on-track with itself, $(f(T_1), v_0v_1)$, (h, v_2v_3), (g, e), (f, v_6v_7) and $(f(T_5), v_8v_9)$, but not with (g, v_2v_3), which is on-track with (f, v_8v_9). An identifier like f near a cycle edge indicates the face of H that is incident to this cycle edge (hence, T_4 is non-proper). While $(f(T_1), v_0v_1)$, (h, v_2v_3) and (g, e) are transfer pairs of T (depicted by red arrows throughout this paper), (f, v_6v_7) and $(f(T_5), v_8v_9)$ are not: the former, as neither v_6v_7 nor v_7v_8 is an extremal C-edge of f; the latter, as T_5 has only one opposite face. The transfer arches of T are T_1, T_2 and T_3; we always color arches that are known to be transfer arches *gray*.

In order to describe the weight transfers through tunnels, we define the following reflexive and symmetric relation for (not necessarily different) faces g and g' of H and edges e and e' of C. Let (g, e) be *on-track with* (g', e') if (see Figure 2)

- e' is incident to g', and e is incident to g,
- there is an acyclic track $T = (T_1, \dots, T_k)$ and $1 \leq i \leq j \leq k$ such that e' is an extremal C-edge of T_i, e is an extremal C-edge of T_j, $j - i$ is minimal, and the following statements are equivalent:
  - g and g' are contained in the same region of $\mathbb{R}^2 - C$
  - either $i = j$ and $e = e'$ or $j - i$ is odd.

Clearly, this relation is an equivalence relation. Note in the definition above that $m_{T_1,T_k} = 2$ may occur and that, if (g, e) is on-track with (g', e'), the set $\{T_i, T_j\}$ is uniquely determined. Moreover, if e is an extremal C-edge of a 3-arch A of a tunnel T, $(f(A), e)$ is on-track with exactly one exit pair of T. Tunnels will serve as objects through which we can pull weight over long distances. We will later prove that tunnels transfer weights only one-way, i.e. towards the exit face of at most one of its tracks. Based on the structure of G, this weight may not be transferred through the whole track; the following definition restricts the parts of the track where weight transfers may occur.

Let T be an acyclic track, e an extremal C-edge of an arch B of T such that $(g, e) := (f(B), e)$ is on-track with the exit pair of T, b the extremal C-edge of B different from e, and h the b-opposite face of g (see Figure 2). Informally, (h, b) is the pair on-track with (g, e) in T that precedes (g, e). Recursively, we define that (g, e) is a *transfer pair* of T if (h, b) is either the exit pair of T or a transfer pair, and

- g is thick,
- the e-opposite face f of g is minor, $m_f \geq 3$, and $h \neq f$, and
- e is either an extremal C-edge of g or adjacent to such an edge, and in the latter case the middle C-edge of B is either
  - incident to f, or
  - incident to a major face such that e is an extremal C-edge of a 3-arch $A \neq B$ whose other extremal C-edge is not incident to a major face.

An arch B of a track T is called *transfer arch* of T if $(f(B), e)$ is a transfer pair of T, where e is the extremal C-edge of B such that $(f(B), e)$ is on-track with the exit pair of T. Note that a transfer arch of T is not necessarily a transfer arch of the other track of the tunnel. A *transfer arch of a tunnel* is an arch that is a transfer arch of at least one of the tracks of the tunnel.

3.2 Discharging Rule

By saying that a face g *pulls weight x over* its C-edge e for a weight x, we mean that x is added to g and subtracted from the e-opposite face of g; we sometimes omit x if the precise value is not important, but positive.

Definition 8 (Discharging Rule)**.** For every minor face g of H and every C-edge e of g (both in arbitrary order), g pulls weight 1 over e from the e-opposite face f of g for every of the following conditions that is satisfied (see Figure 3 for visual explanations of the cases).

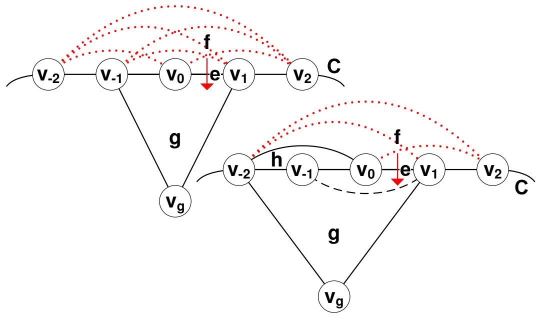
C1: f is major

C2: f is minor, and g is either a thick 2-face or a 3-face whose middle C-edge is incident to a thin minor 2-face $h \neq f$

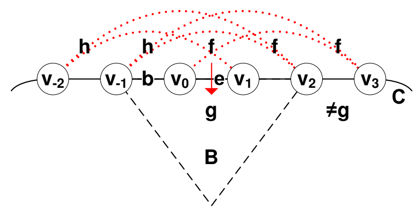
C3: f is minor and $m_f \geq 3$, e is the middle C-edge of a 3-arch B of g and not an extremal C-edge of a 3-arch of f, B has no opposite major face, an extremal C-edge of B is an extremal C-edge of g, and we have that the other extremal C-edge of B is incident to a face $h \notin \{f, g\}$ or $m_g = 3$

C4: f is minor, e is a non-extremal C-edge of a 4-arch B of g such that the extremal C-edge of B that is adjacent to e is an extremal C-edge of g, the other extremal C-edge of B is incident to a thick minor 2-face h, and $m_{f,B} = 3$

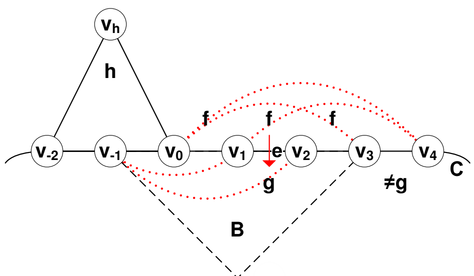
C5: f is minor, e is a non-extremal C-edge of a 4-arch B of g such that the extremal C-edge of B that is adjacent to e is an extremal C-edge of g, the other extremal C-edge b of B satisfies that (h, b) is a transfer pair, the extremal C-vertex of B that is incident to b is not an extremal C-vertex of a 2-arch of g or of h, e is not an extremal C-edge of a 3-arch of f or of g, and $m_{f,B} = 3$



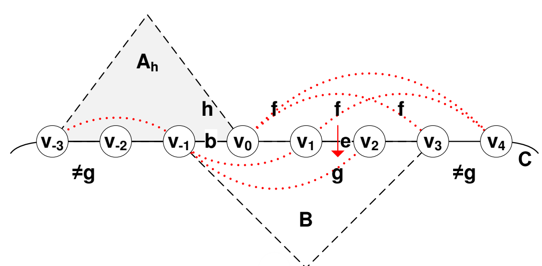
(a) Condition C2: g is either a thick 2-face or a 3-face whose middle C-edge is incident to a thin minor 2-face $h \neq f$. The arrows depict that g pulls weight 1 over e; we do not indicate weights pulled over other edges here. The vertex v_g of a minor face g is drawn only if (as here) g is known to be thick. The *red dotted* arches do not exist in G throughout this paper.



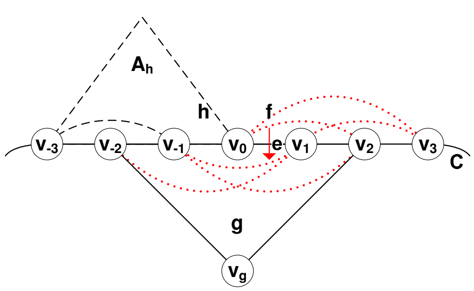
(b) Condition C3: $m_f \geq 3$, B has no opposite major face, e is not an extremal C-edge of a 3-arch of f, v_1v_2 is an extremal C-edge of g, and we have that b is incident to a face $h \notin \{f, g\}$ or $m_g = 3$. Arches like B that are not known to be proper (i.e. that are not known to be in H) are drawn *dashed* throughout this paper. Note that g may have more than three C-edges.



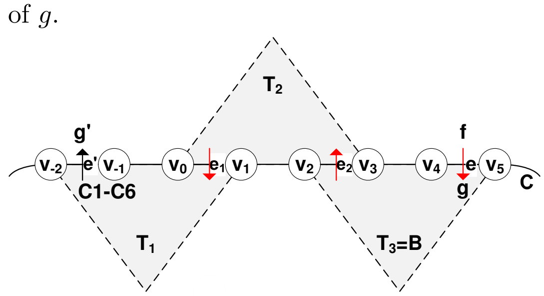
(c) Condition C4.



(d) Condition C5: (h, b) is a transfer pair, no 2-arch of g or h has extremal C-vertex v_{-1}, and e is not an extremal C-edge of a 3-arch of f or of g.



(e) Condition C6: $h \neq f$, and e is not the middle C-edge of a 3-arch of g.



(f) Condition C7: (g, e) is a transfer pair of the acyclic track (T_1, T_2, T_3), and (g', e') satisfies at least one of the conditions $C1$–$C6$.

Figure 3: Conditions C2–C7

C6: f is minor, e is a non-extremal C-edge of a thick 4-face g such that the extremal C-edge of g that is not adjacent to e is the middle C-edge of a 3-arch A_h of a face $h \neq f$, and e is not the middle C-edge of a 3-arch of g

C7: f is minor, (g, e) is a transfer pair of an acyclic track T, and the exit pair (g', e') of T satisfies (in the notation g and e) at least one of the conditions $C1$–$C6$

Note that the weight transfers of this rule are solely dependent on G and C (and not on the current weight transfers). In particular, this holds for the ones caused by $C7$, as these do not depend on other transfers caused by $C7$, and the ones caused by $C5$, as these are only dependent on transfer pairs, not transfers. Since tunnels partition a subset of 3-arches, it suffices to evaluate $C7$ once for each track after Conditions $C1$–$C6$ have been evaluated.

After the discharging rule has been applied, $C7$ effectively routes weight 1 through a part of an acyclic track towards its exit face if this exit face pulls weight from T by any other condition. By definition of $C1$–$C6$, the only faces that do not have an arch of a tunnel T (i.e. reside "outside" T) but pull weight over a C-edge of such an arch are the exit faces of T; in this sense, weight may leave T only through an exit face of T.

3.3 Structure of Tunnels and Transfers

We give further insights into the structure of tunnels and the location of edges over which our discharging rule pulls (positive) weight.

Lemma 9. *Every track $(T_1, \ldots, T_k)$ with $k \geq 3$ satisfies $m_{T_1,T_k} = 0$. In particular, every tunnel is acyclic.*

Proof. Assume first that $(T_1, \ldots, T_k)$ is cyclic, i.e. that $m_{T_1,T_k} = 1$. Since $k \geq 3$, this implies $c = 2k$. Consider the middle C-edge e of any T_i and let g be the e-opposite face of $f(T_i)$. If g is minor, g is the face of some $T_j \neq T_i$ of T, as otherwise g would be a minor 1-face of H, which contradicts our assumption that H has no minor 1-face. Hence, H has at most k minor faces, so that Inequality (1) holds. This implies $c \geq \frac{2}{3}(n+4)$ by Lemma 7, which contradicts our assumption $c < \frac{2}{3}(n+3)$.

Hence, $m_{T_1,T_k} \neq 1$. No two 3-arches have the same set of extremal C-vertices, as such a set would be a 2-separator of G, which contradicts that G is polyhedral. Thus, $m_{T_1,T_k} \in \{0, 2\}$. Assume to the contrary that $m_{T_1,T_k} = 2$. Since $k \geq 3$, this implies $c = 2k - 1$. As G is polyhedral, the C-vertex v_1 of T_1 (in the notation of Figure 4) has degree at least three in G. Thus, $v_{-1}v_1 \in E(G)$ or $v_1v_3 \in E(G)$, say the latter by symmetry. This implies that the face f of T_1 is thick, so that the vertex v_f exists. If f is a 3-face (i.e. T_1 is proper), C is extendable, as the cycle obtained from C by replacing the path $v_{-1}v_0v_1v_2v_3$ with the path $v_{-1}T_kv_2v_1v_0v_fv_3$ shows (this adds one or two new vertices to C).

Since this contradicts our assumption, $m_f \geq 4$. If $m_f = 4$, v_3v_4 is a C-edge of f, since G is plane. Then C is extendable by the path replacement shown in Figure 4. In the remaining case $m_f \geq 5$, planarity implies that T_3 or T_{k-1} is a 3-arch of f. Then H has at most $k - 1$ minor faces, so that Inequality (1) holds, which contradicts our assumptions due to Lemma 7. □

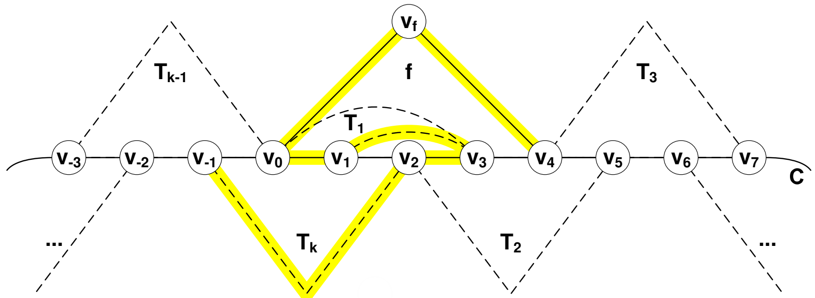


Figure 4: $m_f = 4$: The fat yellow path replacement shows that C is extendable.

We remark that it is possible, but more involved, to prove Lemma 9 solely by using the discharging rule of Definition 8. Using Lemma 9, we assume from now on that every tunnel is acyclic. We next show that G does not contain the dotted arches of Figure 3 for the respective conditions; this sheds first light on the implications that are triggered by the assumption that C is not extendable.

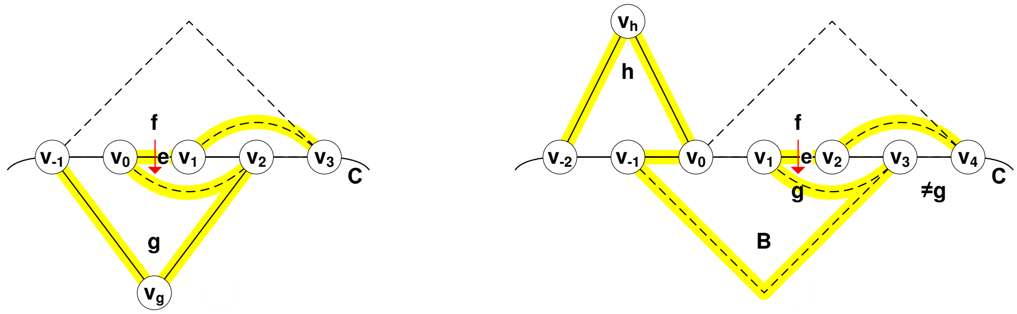
Lemma 10. *For any satisfied condition $X \in \{C2, \dots, C6\}$, none of the red dotted arches in the respective Figure 3a–3e exist in G. If $X = C2$ and g is a 3-face, then $v_{-1}v_1 \in E(G)$; if $X = C6$, then $v_{-3}v_{-1} \in E(G)$.*

Proof. We use the notation of Figure 3. Assume $X = C2$. First, let g be a 2-face in Figure 3a and recall the definition of e and f of Definition 8. If v_0v_1 (or, by symmetry, $v_{-1}v_0$) is a C-edge of a 2-arch A of f, v_0 is an extremal C-vertex of A, since $\{v_{-1}, v_1\}$ is not a 2-separator of G; then C is extendable by the path replacement $v_{-1}v_gv_1v_0Av_2$ (this adds one or two new vertices to C, depending on whether A is proper and thick), which contradicts our assumptions. If v_0v_1 (or, by symmetry, $v_{-1}v_0$) is the middle C-edge of a 3-arch, we have $v_0v_2 \in E(G)$, as G is polyhedral and thus $\deg_G(v_0) \geq 3$; since this contradicts the previous result, neither v_0v_1 nor $v_{-1}v_0$ is the middle C-edge of a 3-arch. Using the same argument, v_0 is not the middle C-vertex of any 4-arch.

Let g be a 3-face. Since G is polyhedral, we have $\deg_G(v_{-1}) \geq 3$ and thus $v_{-1}v_1 \in E(G)$ by planarity. Since h is thin, g is thick. If $v_0v_2 \in E(G)$, C is extendable by the path replacement $v_{-2}v_gv_1v_{-1}v_0Av_2$ (adding exactly one new vertex to C). Since h is thin, f is thin, so that $v_0v_2 \notin E(G)$ implies that no arch has extremal C-vertices v_0 and v_2. Since neither $\{v_{-2}, v_1\}$ nor $\{v_{-2}, v_2\}$ is a 2-separator of G and $v_0v_2 \notin E(G)$, no face different from g has an arch whose set of extremal C-vertices is $\{v_{-2}, v_1\}$ or $\{v_{-2}, v_2\}$.

Assume $X = C3$. By definition of $C3$, v_0v_1 is not an extremal C-edge of a 3-arch of f. Since G is polyhedral, v_0v_1 is not the middle C-edge of a 3-arch of f. Assume to the contrary that v_1 (or v_0 by a symmetric argument) is the middle C-vertex of a 4-arch of f. Then $v_{-1}v_0$ is incident to f, which implies by $C3$ that g is a minor 3-face. Because $\{v_{-1}, v_2\}$ is not a 2-separator of G, a vertex of $\{v_0, v_1\}$ is adjacent to v_3 in G. Since $v_0v_3 \notin E(G)$, we have $v_1v_3 \in E(G)$ and thus $v_0v_2 \in E(G)$ by $\deg_G(v_0) \geq 3$. Then g is thick and C is extendable by Figure 5a.

Assume $X = C4$. Then the first result of the case $X = C2$ implies $v_{-1}v_1 \notin E(G)$. In addition, $v_{-1}v_2 \notin E(G)$, as otherwise C is extendable by the path replacement



(a) $C3$ when v_1 is the middle C-vertex of a 4-arch of f.

(b) $C4$ when v_2 is the middle C-vertex of a 4-arch of f.

Figure 5: Restrictions of $C3$ and $C4$

$v_{-2}v_hv_0v_1v_2v_{-1}Bv_3$. Hence, v_1v_2 is not an extremal C-edge of a 3-arch of g. If v_1v_2 is an extremal C-edge of a 3-arch A of f, C is extendable by the path replacement $v_{-2}v_hv_0v_{-1}Bv_3v_2v_1Av_4$, as this adds at most three new vertices to C. Note that, if A is proper, we have $v_A \neq v_h$ in this replacement (A is thick, because h is), as H has no minor 1-face. In addition, v_1v_2 is not the middle C-edge of a 3-arch of f, as otherwise $\{v_0, v_3\}$ would be a 2-separator of G by the previous results. If v_2 is the middle C-vertex of a 4-arch of f, we have $v_2v_4 \in E(G)$, as otherwise $\{v_0, v_3\}$ would be a 2-separator of G. Since $\deg_G(v_1) \geq 3$, this implies $v_1v_3 \in E(G)$, so that C is extendable by Figure 5b.

Assume $X = C5$. By definition of transfer arches, $4 \leq m_g \leq 5$. If v_1v_2 is the middle C-edge of a 3-arch of f, $\{v_0, v_3\}$ is a 2-separator of G, since $v_{-1}v_1$ and $v_{-1}v_2$ are not contained in G. This contradicts that G is polyhedral. Assume to the contrary that v_2 is the middle C-vertex of a 4-arch A of f. If $m_g = 4$, C is extendable by the replacement $v_{-1}v_gv_3v_2v_1v_0Av_4$, so let $m_g = 5$. Then $v_2v_4 \notin E(G)$, as otherwise C is extendable by the replacement $v_{-2}v_gv_3v_{-1}v_0v_1v_2v_4$. Then $\{v_0, v_3\}$ is a 2-separator of G, which contradicts that G is polyhedral.

Assume $X = C6$. Then $v_{-1}v_1 \notin E(G)$, as otherwise C is extendable by the replacement $v_{-3}A_hv_0v_1v_{-1}v_{-2}v_gv_2$, as v_g exists since g is thick. Since G is polyhedral, $\deg_G(v_{-1}) \geq 3$, which implies $v_{-3}v_{-1} \in E(G)$ as only remaining option. Then $v_{-2}v_1 \notin E(G)$, as otherwise C is extendable by $v_{-3}v_{-1}v_0v_1v_{-2}v_gv_2$. Since G is polyhedral, this implies that there is no 2-arch of f with middle C-vertex v_1. Assume to the contrary that f has a 2-arch A with middle C-vertex v_2. Then A is thick, as A_h has a 2-arch, and A is not proper, as otherwise f would be a minor 1-face of H due to $h \neq f$. Hence, $v_1v_3 \in E(G)$, so that C is extendable by the replacement $v_{-3}A_hv_0v_{-1}v_{-2}v_gv_2v_1v_3$. In addition, v_1v_2 is not the middle C-edge of a 3-arch, as otherwise v_1 would have degree two in G. □

For a C-edge e of a face g of H and a condition $X \in \{C1, C2, \dots, C7\}$, let $g \overset{e}{\leftarrow} X$ denote that X is satisfied for g and e in Definition 8. For notational convenience throughout this paper, whenever g pulls weight from a face f, we denote by v_0 the extremal C-vertex of f whose clockwise neighbor v_1 in C is a C-vertex of f, and denote by v_i the ith vertex modulo c in a clockwise traversal of C starting at v_1 (see for example Figure 4).

So far, a tunnel might transfer weights through both of its tracks simultaneously. The next lemma shows that this never happens.

Lemma 11. *Let (g,e) and (g',e') be the exit pairs of a tunnel T such that g pulls weight over e. Without loss of generality, assume that (g,e) is the counterclockwise exit pair of T, let $(T_1,\dots,T_k)$ be the counterclockwise track of T and let T_0 be the proper arch of g. Then the following statements hold.*

(i) *g is minor, $g\overset{e}{\leftarrow}C2$ and no other condition of $C1$–$C7$ is satisfied for (g,e)*

(ii) *Let $P := T_i v_{2i+1} v_{2i} T_{i+1} v_{2i+3} v_{2i+2} \dots T_j$ be a non-simple subpath of a path replacement such that*

- *$0 \le i < j \le k$ and*
- *for every $i+2 \le x < j$, every 2-arch A of T_x has a C-edge b such that $(f(A),b)$ is on-track with (g,e) (confer (iii)).*

Then P can be modified into a subpath that is simple, uses a subset of its original vertices and whose positive amount of new vertices is at most one plus the number of faces of size five in G that are enclosed by both $T_i \cup \dots \cup T_j$ and C.

(iii) *every 2-arch A of an arch of T has a C-edge b such that $(f(A),b)$ is on-track with (g,e)*

(iv) *$g \neq g'$ and there is no 2-arch of g' that has C-edge e'*

(v) *g' does not pull any weight over e'*

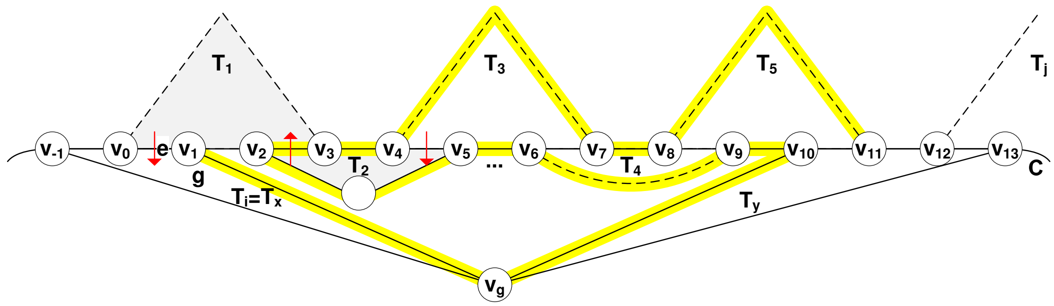
(vi) *for every 4-arch A that has an arch T_i of T, the common extremal C-edge b of A and T_i satisfies that $(f(A),b)$ is on-track with (g,e)*

(vii) *every arch T_i of T that is consecutive to two transfer arches of T satisfies $m_{f(T_i)} \le 4$*

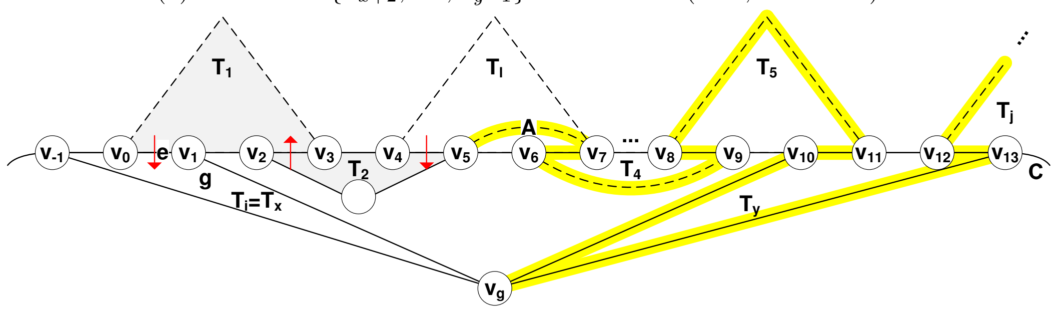
Proof. Let X be a condition in $\{C1, C2, \dots, C7\}$ such that $g\overset{e}{\leftarrow}X$. For every $1 \le j \le k$, let e_j denote the edge that joins the two extremal C-vertices of T_j; e_j does not have to be contained in G.

For Claim (i), g is minor, because major faces do not pull weight over any edge. Since $f(T_1)$ is minor by the definition of arches, $X \neq C1$. If $X = C3$, e is an extremal C-edge of the 3-arch T_1, which contradicts the definition of $C3$. If $X \in \{C4, C5, C6\}$, e is an extremal C-edge of the 3-arch T_1 of f, which contradicts Lemma 10 for X. Assume $X = C7$. Then e is an extremal C-edge of a transfer arch A of g, which implies that A and T_1 are consecutive. Hence, A is an arch of the same tunnel T as T_1, so that $A = T_k$ and $k \ge 3$ (the latter due to $c \ge 6$). Hence, T is cyclic, which contradicts Lemma 9 (and the definition of $C7$). We conclude that $X = C2$. By definition of tunnels, the proper arch of g is not part of any tunnel; this holds in particular for the case $m_g = 3$ of $C2$.

For Claim (ii), the only vertices that P may visit more than once are the vertices $v_{f(T_x)}$ for any $i \le x \le j$ such that T_x is proper and thick, as these vertices may be contained in a different proper and thick arch $T_y \neq T_x$. If e_x exists for all such x but one, we obtain the desired simple path by replacing all such T_x with e_x (if necessary, we keep one arch



(a) No arch of $\{T_{x+2},\ldots,T_{y-1}\}$ has a 2-arch (here, $i=x=0$).



(b) An arch T_l of $\{T_{x+2},\ldots,T_{y-1}\}$ has a 2-arch A (here, $i=x=0$).

Figure 6: Dealing with non-simple path replacements.

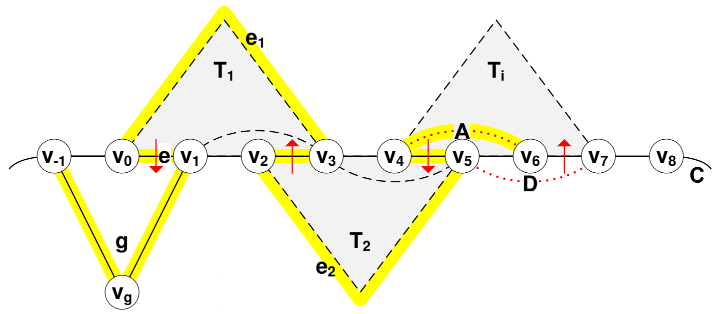
unmodified in order to retain at least one newly added vertex for the path replacement). In the remaining case, there are indices $i \le x < y \le j$ such that $v_{f(T_x)} = v_{f(T_y)}$, both T_x and T_y are proper and thick, neither e_x nor e_y exists, and x is maximal.

By maximality of x, there is no other pair $x \le x' < y' \le y$ than (x,y) itself for which $v_{f(T_{x'})} = v_{f(T_{y'})}$, both $T_{x'}$ and $T_{y'}$ are proper and thick and neither $e_{x'}$ nor $e_{y'}$ exists. Hence, by using e_l whenever possible for every $x+2 \le l < y$, the path replacement shown in Figure 6a is simple; it also adds at least the new vertex $v_{f(T_x)}$. If none of the arches $T_{x+2},\ldots,T_{y-1}$ has a 2-arch, C is therefore extendable by this path replacement, as $f(T_l)$ is a face of size five in G for every $x+2 \le l < y$ such that T_l is minor and thick and e_l does not exist (e.g. for T_2 in Figure 6a). This contradicts our assumption that C is not extendable.

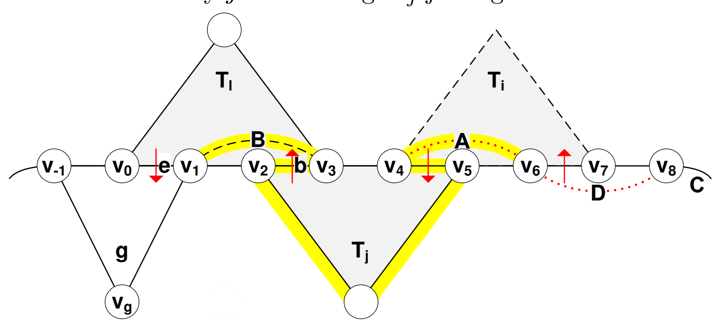
Hence, there is a maximal $x+2 \le l < y$ such that T_l has a 2-arch A. By the second assumption of the claim, A has a C-edge b such that $(f(A),b)$ is on-track with (g,e) (see Figure 6b). Then P can be modified to the subpath shown in Figure 6b, which is simple, adds at least the new vertex $v_{f(T_y)}$, and by maximality of l adds additionally as many new vertices as there are indices $l < z \le y$ such that $f(T_z)$ is a face of size five in G.

For Claim (iii), assume to the contrary that there is a minimal i such that T_i does not satisfy (iii); then T_i has a 2-arch A that has no C-edge b such that $(f(A),b)$ is on-track with (g,e) (see Figure 7a). By Claim (i), $X = C2$. By Lemma 10 (for $C2$), $i \ne 1$.

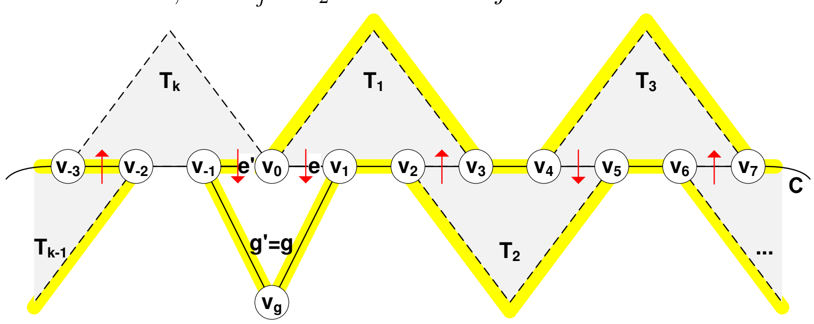
Assume first that, for every $0 < j < i$, G contains the edge e_j. Note that this happens precisely if either $e_j = T_j$ (i.e. T_j is thin or non-proper) or $\{e_j\} \cup T_j$ bounds a triangle whose interior point set is contained in $f(T_j)$ (as G is polyhedral). If A and g do not have



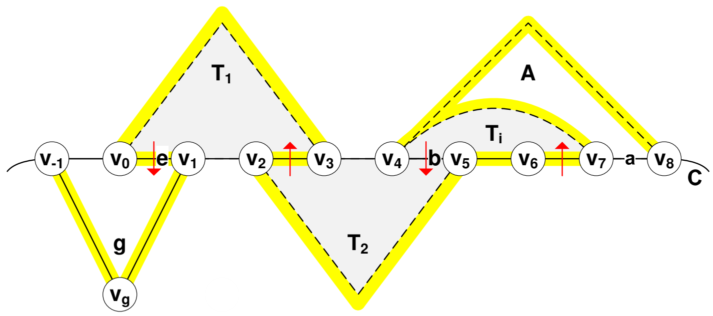
(a) A 2-arch A of T_i that has no C-edge b such that $(f(A), b)$ is on-track with (g, e). Here, the path replacement contains for every $j < i$ the edge e_j joining the two extremal C-vertices of T_j.



(b) $T_l = T_1$ has a 2-arch B, and $T_j = T_2$ has maximal $j < i$ such that G does not contain e_j.



(c) $g = g'$. Here, the fat subgraph depicts the whole cycle C' that replaces C. Note that this case occurs only one step before the desired cycle length is reached.



(d) A 4-arch A having an arch $T_i \in T$ such that $(f(A), b)$ is not on-track with (g, e).

Figure 7: Exit face g pulling weight from a tunnel such that $m_g = 2$. For the case $m_g = 3$, $v_{-1}v_gv_1v_0$ is replaced with $v_{-2}v_gv_1v_{-1}v_0$.

any C-vertex in common, C is extendable by the path replacement shown in Figure 7a, as this path is simple and adds exactly one new vertex to C (namely, v_g). Otherwise, A and g have a common C-vertex, which is not possible in the case $m_g = 3$ of $C2$ because of T_i, as h is proper in that case. Hence, $m_g = 2$ and, by Lemma 10 (for $C2$), A and g have exactly v_{-1} in Figure 7a as common C-vertex, which implies $i = k$ and $g = g'$. Then the same replacement (this time specifying the whole cycle that replaces C) shows that C is extendable.

In the remaining case, T has an arch T_j such that $j < i$ is maximal and G does not contain e_j; in particular, T_j is thick and proper, so that $v_{f(T_j)}$ exists. Define $T_0 := g$ and let $0 \leq l \leq j$ be maximal such that T_l has a 2-arch B (see Figure 7b). Such l exists, as $l = 0$ is a valid choice for the case $m_g = 2$ and, by Lemma 10, also for the case $m_g = 3$. By minimality of i, B has a C-edge b such that $(f(B), b)$ is on-track with (g, e) (if $l = 0$, we have $(f(B), b) = (g, e)$).

Consider the path replacement of Figure 7b, which contains besides edges of C only A, B, and the arches T_z and edges e_z that satisfy $l < z < i$. By minimality of i, all 2-arches of $T_l, \dots, T_{i-1}$ satisfy the second condition of Claim (ii), so that this path replacement can be made simple by applying Claim (ii) to the subpath $T_l \dots T_{i-1}$. For every $j < z < i$, the maximality of j implies that G contains e_z, so that using e_z in the replacement does not add any new vertex to C.

For every $l < z < j$, we use the edge e_z in the replacement if G contains it; if so, this does not add a new vertex to C. If otherwise $e_z \notin E(G)$, T_z is thick and proper and has no 2-arch by maximality of l, so that $f(T_z)$ is a face of size five in G. If $l < j$, T_j has no 2-arch, so that $f(T_j)$ is a face of size five in G for the same reason. We conclude that, if $l < j$, the path replacement adds a positive amount of at most $1 + n_5(G)$ (1 because of g) new vertices to C, so that C is extendable by applying Claim (ii) to the path. If otherwise $l = j$, C is extendable by restricting the replacement to $v_2 T_j v_5 B v_3 v_4 e_{j+1} \dots e_{i-1} v_5 v_4 A$, which adds again a positive amount of at most 1 (because of T_j) new vertex to C, and applying Claim (ii) to this restriction.

For the first statement of Claim (iv), assume to the contrary that $g = g'$ (see Figure 7c). By Claim (i), $X = C2$. Since $g = g'$ pulls weight over both edges e and e' due to $C2$, we may apply Claim (iii) for both exit pairs of T. This implies that no arch of T has a 2-arch. Consider the fat cycle of Figure 7c that replaces C by omitting an arbitrary arch of T (in Figure 7c, T_k is omitted). If this cycle is not simple, there are indices $0 \leq x < y \leq k$ such that $v_{f(T_x)} = v_{f(T_y)}$, both T_x and T_y are proper and thick, neither e_x nor e_y exists, and x is maximal; then C is extendable by the simple path replacement of Figure 6a, as this adds a positive amount of at most $1 + n_5(G)$ new vertices to C. In the remaining case, the cycle of Figure 7c is simple. Then for every arch T_z that is not omitted in this replacement, either G contains e_z or $f(T_z)$ is a face of size five in G. Hence, replacing C with this cycle adds a positive amount of at most $1 + n_5(G)$ (1 because of g) new vertices to C, so that C is extendable.

For the second statement of Claim (iv), assume to the contrary that there is a 2-arch D of g' that has C-edge e'. If $m_{D,T_k} = 2$ (see Figure 7a when $i = k$), there is a 2-arch A of T_k that contradicts Claim (iii), as G has no vertex of degree two. Hence, $m_{D,T_k} = 1$ (see D in Figure 7b when $i = k$). If D has a common C-edge with g, we have $m_g = 2$ by planarity and $m_{D,g} \neq 1$ by Lemma 10; this implies $g = g'$, which contradicts the previous claim. In the remaining case, D and g have no common C-edge. Then C is extendable by the same path replacements of Figures 7a and 7b as in Claim (iii), except that A is possibly replaced by D. This adds a positive amount of at most $2 + n_5(G)$ new vertices to C (i.e. at most one more as in Claim (iii)), because D may be thick and proper.

For Claim (v), assume to the contrary that there is a condition $Y \in \{C1, \dots, C7\}$ such that $g' \overset{e'}{\leftarrow} Y$. By Claim (i) (for both $g \overset{e}{\leftarrow} X$ and $g' \overset{e'}{\leftarrow} Y$), $X = Y = C2$. This contradicts Claim (iv) in both cases $m_g = 2$ and $m_g = 3$.

For Claim (vi), assume to the contrary that a 4-arch A has some $T_i \in T$ such that the common extremal C-edge b of A and T_i does not satisfy that $(f(A), b)$ is on-track with (g, e) (see Figure 7d). Let a be the C-edge of A that is not a C-edge of T_i. Assume first that a is a C-edge of g. Then $m_g = 2$, $i = k$, $c = 2k + 2$ and e' is not incident to a minor 2-face by Claim (iv). Since H has no minor 1-face, H has thus at most $k + 1$ minor faces, so that Inequality (1) holds. This implies $c \geq \frac{2}{3}(n + 4)$ by Lemma 7, which contradicts our assumptions. Hence, a is not a C-edge of g.

Consider the replacement of Figure 7d and note that this replacement is also valid when A and g share exactly one C-vertex. If G contains e_j for every $j < i$, C is extendable by this replacement, as this adds at most 2 new vertices to C. Otherwise, we may proceed as in the proof of Claim (iii) and compensate the usage of new vertices that are added to C with faces of size five of G, so that at most $2 + n_5(G)$ (2 because of A and either T_j or g) new vertices are used. By Claim (iii), we may apply Claim (ii) to ensure that all path replacements are simple, so that C is extendable.

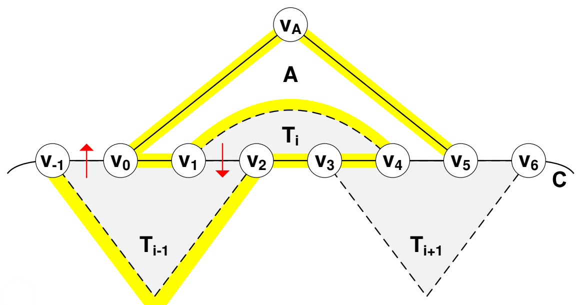


Figure 8: A 5-arch A having exactly the C-edges of T_i as non-extremal C-edges.

For Claim (vii), assume to the contrary that T contains an arch $T_i \notin \{T_1, T_k\}$ with minimal i and $m_{f(T_i)} \geq 5$ such that T_{i-1} and T_{i+1} are transfer arches. If every C-edge of T_{i+1} is incident to $f(T_i)$, T_{i+1} contradicts the definition of transfer arches, as all its C-edges are opposite to the same face. By the same argument, not every C-edge of T_{i-1} is incident to $f(T_i)$. We conclude that $m_{f(T_i)} = 5$ and that the C-edges of T_i are exactly the non-extremal C-edges of $f(T_i)$. Since T_i is an arch, $f(T_i)$ is minor. Since T_i is a 3-arch of $f(T_i)$ and $m_{f(T_i)} = 5$, $f(T_i)$ is thick. Hence, $v_{f(T_i)}$ exists and C is extendable by Figure 8. □

By Lemma 11(v), all weight transfers that are caused within a tunnel by Condition $C7$ are one-way, i.e. use only one track of T. As an immediate implication, the following lemma shows that all weight transfers strictly within a tunnel are solely dependent on the weight transfers on its exit pairs.

Lemma 12. *Let (g, e) be a transfer pair of a track T with exit pair (g', e'). Then g pulls weight over e if and only if g' pulls weight over e' (and if so, $g \overset{e}{\leftarrow} C7$ and $g' \overset{e'}{\leftarrow} C2$ such that $m_{g'} = 2$).*

Proof. Assume that g' pulls weight over e'. By Lemma 9, T is acyclic. By Lemma 11(i), $g' \overset{e'}{\leftarrow} C2$; in particular, $2 \leq m_{g'} \leq 3$. Hence, $C7$ is satisfied for (g, e), so that g pulls weight over e. Assume to the contrary that $m_{g'} = 3$ and let B be the first 3-arch of T. Then B is thin by definition of $C2$. Thus, B is no transfer arch of T, which contradicts that (g, e) is a transfer pair of T. Hence, $m_{g'} = 2$.

Assume to the contrary that $g \overset{e}{\leftarrow} X$ for some $X \in \{C1, \dots, C7\}$ and g' does not pull any weight over e'. The latter implies $X \neq C7$. Since (g, e) is a transfer pair, the e-opposite face of g is minor. Hence, $X \neq C1$. Since e is a C-edge of a 3-arch B of T with $f(B) = g$, $m_g \geq 3$; as B is contained in a tunnel, g is not a minor 3-face whose middle C-edge is incident to a thin minor 2-face. Hence, $X \neq C2$. By planarity, $X \neq C3$. Since e is an extremal C-edge of B, Lemma 10 implies $X \notin \{C4, C5, C6\}$, which is a contradiction. □

An immediate implication of the discharging rule in Definition 8 is that faces pull only integer weights over edges, as every satisfied condition adds 1 to that weight. We next prove that no two of the conditions $C1$–$C7$ are satisfied simultaneously for the same face g and edge e; hence, the amount of weight that g pulls over e is either weight 0 or 1. This is crucial for keeping the amount of upcoming arguments on a maintainable level; in fact, our conditions were designed that way.

Lemma 13. *The total weight pulled by a face of H over its C-edge e is either 0 or 1. If it is 1, the e-opposite face does not pull any weight over e.*

Proof. Assume to the contrary that e is incident to two faces f and g of H such that $f \overset{e}{\leftarrow} X$ and $l \overset{e}{\leftarrow} Y$ for conditions X and Y and $l \in \{f, g\}$, and such that $X = Y$ implies $l = g$ (as otherwise a single pull is counted twice). Without loss of generality, we assume that Y is not stated before X in Definition 8. If $X = C1$, g is major, which implies $Y = C1$ and thus $l = g$; then f is major, which contradicts $f \overset{e}{\leftarrow} X$. Hence, $X \neq C1$, so that both f and g are minor.

If $Y = C7$, e is an extremal C-edge of a 3-arch. Then $X \notin \{C3, \dots, C6\}$ by planarity and Lemma 10 for each $l \in \{f, g\}$: for example, $X = C3$ identifies e in Figure 3b with edge e of Figure 3f while l pulls weight over e by $C7$, so that $l = g$ contradicts planarity, and $l = f$ contradicts that a red dotted 3-arch does not exist by Lemma 10. We also have $X \neq C7$ by Lemmas 11(v) and 12, so that $X = C2$. Since minor 3-faces whose middle C-edge is incident to a thin minor 2-face are not contained in any tunnel, we have $l = g$. By $C2$, e is a C-edge of a 2-arch of f, which contradicts Lemma 11(iii) or (iv). We conclude that $Y \neq C7$, which implies $X \neq C7$. We distinguish the remaining options for X and Y in $\{C2, \dots, C6\}$.

Assume $X = C2$. If $m_f = 3$, e is an extremal C-edge of a 3-face, so that Lemma 10 and planarity imply $Y \notin \{C3, C4, C5, C6\}$ for each $l \in \{f, g\}$. Hence, $m_f = 2$. Then $l = g$, as the remaining options $Y \in \{C3, C4, C5, C6\}$ for $l = f$ require $m_f \geq 3$. By Lemma 10 (for $C2$ and $C6$), $Y \notin \{C2, C6\}$. In the remaining case, $Y \in \{C3, C4, C5\}$ contradicts $m_f = 2$.

Assume $X = C3$. Then e is the middle C-edge of a 3-arch A of f, so that $l = g$ implies $Y \notin \{C3, C4, C5, C6\}$ by Lemma 10. We conclude $l = f$. If $Y \in \{C4, C5\}$, g is the only opposite face of A, which implies by $C3$ that f is a 3-face; this contradicts that f has a 4-arch. Thus $Y = C6$, which contradicts Lemma 10 (for $C6$).

Assume $X \in \{C4, C5\}$. If $l = f$, $Y \notin \{C5, C6\}$, as 2-faces do not have 3-arches, so let $l = g$. Then $Y \in \{C4, C5\}$ contradicts Lemma 10 (for Y) and $Y = C6$ contradicts planarity.

Assume $X = C6$. Then $Y = C6$ and thus $l = g$, which contradicts that G is plane. □

By Lemma 13, we know that whenever weight 1 is pulled over an edge e by some condition $C1$–$C7$, no other condition is satisfied on e, so that 1 is the final amount of weight transferred over e.

3.4 The Proof

Initially, we charge every j-face of H with weight j. Throughout this section, let w denote the weight function on the set of faces of H after our discharging rule has been applied. Clearly, $\sum_{f \in F(H)} w(f) = 2c$ still holds. In order to prove Inequality (1) if $V^- \neq \emptyset$ and otherwise Inequality (2), we aim to show that every minor face f satisfies $w(f) \geq 4$ if f is thick and $w(f) \geq 2$ if f is thin such that no face has negative weight.

For $S \subseteq E(C)$ and the set Z of C-edges of a face f of H, let the (weight) *contribution* of S to f be $|S \cap Z|$ (i.e. the initial weight the edges of S give to $w(f)$) plus the sum of all weights pulled by f over edges in $S \cap Z$ minus the sum of all weights pulled by opposite faces of f over edges in $S \cap Z$. The *contribution* of an arch A to f is the contribution of its C-edges to f; this way, every proper arch A contributes weight $w(f(A))$ to $f(A)$. Since f loses weight at most 1 over every of its C-edges by Lemma 13, we have $w(f) \geq z$ if a set S contributes weight z to f.

The *distance* of two edges in a connected graph is -1 plus the length of a shortest path containing both edges. By our discharging rule, most pulls occur on C-edges that are extremal or adjacent to an extremal C-edge; the following definition captures the remaining pulls and will be used in a final counting argument. For a condition $X \in \{C1, C2, \ldots, C7\}$, let $g \overset{e}{\leftarrow} X$ be a *mono-pull* if $X = C3$ and e and its two adjacent edges in C are incident to a common face $f \neq g$. An edge $e \in E(C)$ is *mono* if it is incident to a face g such that $g \overset{e}{\leftarrow} C3$ is satisfied and a mono-pull, and *non-mono* otherwise.

Lemma 14. *For two C-edges e and b of a minor face f of H and conditions X and Y, let $g \overset{e}{\leftarrow} X$ and $h \overset{b}{\leftarrow} Y$ such that $f \notin \{g, h\}$ and X and Y are not contained in $\{C2, C7\}$. Then*

(i) if $g \overset{e}{\leftarrow} X$ is no mono-pull, e is either an extremal C-edge of f or adjacent to one (more precisely, an extremal C-edge of f if and only if $X \in \{C3, C6\}$), and

(ii) e and b have distance at least three in C.

Proof. Consider Claim (i). Since f is minor, $X \neq C1$. If $X = C3$, e is an extremal C-edge of f, since $g \overset{e}{\leftarrow} X$ is no mono-pull. For every $X \in \{C4, C5\}$, e is adjacent to an extremal

C-edge of f by definition of X (for $X = C5$, this follows from the transfer pair). If $X = C6$, e is an extremal C-edge of f.

For Claim (ii), assume to the contrary that e and b have distance at most two in C. Since f is minor, $C1 \notin \{X, Y\}$. Let $X = C3$ and let B be the 3-arch of g that has middle C-edge e. Then the existence of B and planarity imply $Y \notin \{C3, C4, C5\}$, and Lemma 10 and planarity imply $Y \neq C6$. Hence, $X \in \{C4, C5, C6\}$ and, by symmetry, the same holds for Y. Then $Y \notin \{C4, C5, C6\}$ by planarity, Lemma 10 and the respective condition Y imposes on f. This is a contradiction. □

Lemma 15. *Let $g \overset{e}{\leftarrow} X$, f be the e-opposite face of g, B be the arch of g shown in Figure 3 (for $X = C6$, let B be the proper arch of g), and S be the set of common C-edges of f and B.*

- *If $X = C3$ and $|S| = 3$ (i.e. $g \overset{e}{\leftarrow} X$ is a mono-pull), each of the two extremal C-edges of B contributes weight at least 1 to f.*
- *If $X \in \{C4, C5\}$, S contributes weight at least 2 to f.*
- *If $X = C6$, S contributes weight at least 1 to f.*

Proof. For every $X \in \{C3, \dots, C6\}$, f is minor. Assume $X = C3$ and $|S| = 3$. Then every C-edge of B is incident to f, which implies $m_g = 3$ (by $C3$). Assume to the contrary that S contributes weight at most 1 to f. Then $g \overset{b}{\leftarrow} Y$ for a C-edge $b \neq e$ of B and some condition Y by Lemma 13. Since f is minor, $Y \neq C1$. By $m_g \geq 3$ and the definition of $C2$ in that case, $Y \neq C2$. By Lemma 14(ii), $Y \notin \{C3, C4, C5, C6\}$. Hence, $Y = C7$. Then (g, b) is a transfer pair, which contradicts that every C-edge of B is incident to f.

Let $X \in \{C4, C5, C6\}$. Then $m_g \geq 4$, $|S| = 3$ if $X \in \{C4, C5\}$, and $|S| = 2$ if $X = C6$. Assume to the contrary that g pulls weight over an edge $b \neq e$ of S by some condition Y. Since f is minor, $Y \neq C1$. Since $m_g \geq 4$, $Y \neq C2$. By Lemma 14(ii), $Y \notin \{C3, C4, C5, C6\}$. Hence, $Y = C7$. Then (g, b) is a transfer pair, which contradicts $|S| = 3$ if $X \in \{C4, C5\}$ and Lemma 10 if $X = C6$. □

Lemma 16. *For a minor face f of H, let A be an arch of f with minimal m_A such that a face $h \neq f$ pulls weight over a C-edge b of A by Condition $Y \in \{C2, C7\}$. Then $w(f) \geq 2$ and, if f is thick, $w(f) \geq 4$.*

Proof. Assume that $w(f) < 4$, as otherwise the claim holds. Then $w(f) \leq 3$ by Lemma 13 and, by $C1$, at most one C-edge of f is incident to a major face (we will use this throughout the proof). Since h pulls weight, h is minor. Let A_h be the proper arch of h if $Y = C2$ and the unique 3-arch of h that has C-edge b if $Y = C7$. In both cases, h is thick (for $Y = C7$, as A_h is a transfer arch) and b is an extremal C-edge of A_h. If $Y = C7$, (h, b) is a transfer pair. In that case, let T be the track containing A_h whose exit pair is on-track with (h, b); by Lemma 12 and $h \overset{b}{\leftarrow} Y$, the exit pair of T satisfies $C2$ such that the exit face is a 2-face.

Assume to the contrary that $m_{A,A_h} \geq 2$. If $Y = C2$, this implies $m_h = 2 = m_{A,h}$. Since G has minimum degree at least three, the middle C-vertex of h is then an extremal C-vertex of an arch of A, which contradicts the minimality of m_A. Hence, $Y = C7$. Then $m_{A,A_h} = 2$, as $m_{A,A_h} \neq 3$ by the definition of transfer arches. Consider the non-extremal

C-vertex v of A_h that is incident to b. By minimality of m_A and $\deg_G(v) \geq 3$, v is an extremal C-vertex of a 2-arch of A_h, which contradicts Lemma 11(iii). We conclude that $m_{A,A_h} = 1$; in particular, b is an extremal C-edge of A.

This shows that no opposite face of f pulls weight over a non-extremal C-edge of A using $C2$ or $C7$. We distinguish the following cases for m_A.

Case $m_A = 2$:

Then $Y = C2$ contradicts Lemma 10, and $Y = C7$ contradicts Lemma 11(iii) or (iv) (which one depends on whether there is another arch of T that has C-edge b).

Case $m_A = 3$:

Let v_2v_3 be the extremal C-edge of A that is different from b, and let g be the (possibly major) v_2v_3-opposite face of f, as shown in Figure 9a. If $Y = C2$, v_1 is an extremal C-vertex of h. If $Y = C7$, (h, b) is a transfer pair, so that v_1 or v_2 is an extremal C-vertex of h and $v_{-2}v_{-1}$ is not incident to f. Hence, in all cases, either v_1 or v_2 is an extremal C-vertex of h, which implies $g \neq h$.

By minimality of m_A, $v_0v_2 \notin E(G)$. Since $\deg_G(v_2) \geq 3$ and not both v_1v_2 and v_2v_3 are incident to major faces, v_2 is an extremal C-vertex of an arch whose face is not f. Let B be such an arch with minimal m_B; by the definition of arches, $f(B)$ is minor. Since H has no minor 1-face, $f(B) \in \{g, h\}$. By minimality of m_A, the existence of B and planarity, we know that v_1v_2 is not incident to a thin minor 2-face; hence, A is contained in a tunnel. Let T be the track of this tunnel whose exit pair is on-track with (h, b); note this is consistent with the definition of T in the case $Y = C7$, since A and A_h are consecutive and thus in the same track. By Lemma 12 and $h \xleftarrow{b} Y$, the exit pair of T satisfies $C2$. Thus, $m_B = 2$ contradicts Lemma 11(iii) or (iv). We conclude that $m_B \geq 3$.

We show next that

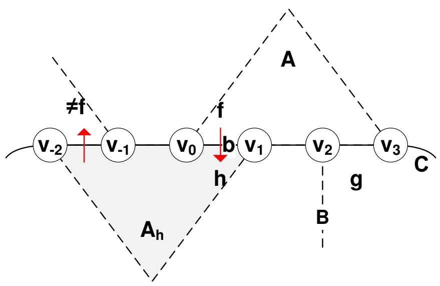
> the weight contribution of $\{v_{-1}v_0, v_1v_2\}$ to f is at least one and, if v_1v_2 is not incident to g, at least two. (*)

By Lemma 10, planarity and the existence of B, the v_1v_2-opposite face of f does not pull weight over v_1v_2 by any condition. This gives the first claim of (*), so assume that v_1v_2 is not incident to g. If v_1v_2 is incident to a major face, the second claim of (*) follows straight from $f \xleftarrow{v_1v_2} C1$. Otherwise, v_1v_2 is incident to $h \neq g$, as H has no minor 1-face. Then $Y = C7$, A_h is non-proper and (h, b) is a transfer pair, so that $v_{-2}v_{-1}$ is not incident to f and $v_{-1}v_0$ is either incident to a major face and g is minor or incident to f. If $v_{-1}v_0$ is incident to a major face and g is minor, the second claim of (*) follows from $f \xleftarrow{v_1v_2} C3$; hence, $v_{-1}v_0$ is incident to f. Then h does not pull weight over $v_{-1}v_0$ using Condition $C3$, as v_1v_2 is incident to h. By Lemma 10, planarity and the existence of B, h does not pull weight over $v_{-1}v_0$ by any other condition, which concludes the proof of (*).

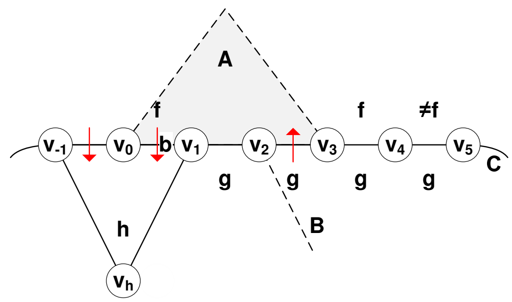
If g is major, $f \xleftarrow{v_2v_3} C1$ and v_1v_2 is not incident to g, which contradicts $w(f) \leq 3$ by (*); hence, g is minor. If $f(B) = g$, $m_B \geq 3$ implies that v_4v_5 is incident to g.

If $f(B) = h$, v_1v_2 is incident to h, so that $Y = C7$ and $m_B \geq m_{A_h} + 1 \geq 4$; then the minimality of m_B implies $m_g \geq 4$, as g is minor. Assume that f is thin; then $m_f = m_A = 3$. If v_1v_2 is not incident to g, (*) gives the claim $w(f) \geq 2$ of the lemma. Otherwise, v_1v_2 is incident to g and, by the existence of B, v_1v_2 is not an extremal C-edge of a 3-arch. Then $f \xleftarrow{v_1v_2} C3$ gives the claim.

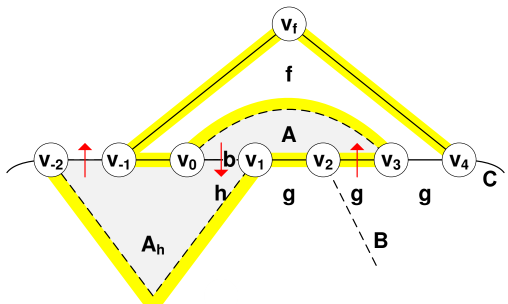
Hence, in the remaining case, f is thick, g is minor, $m_g \geq 3$ and v_4v_5 is incident to g. We distinguish the following cases for T.



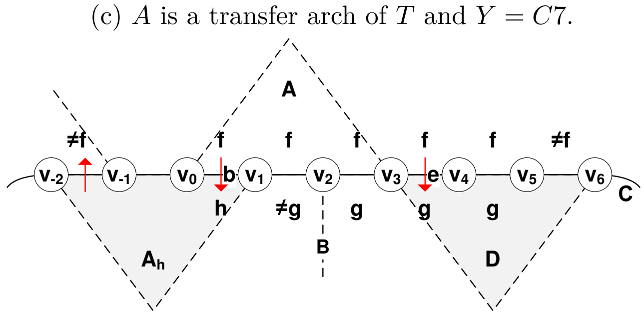
(a) $m_A = 3$ when $Y = C7$.



(b) A is a transfer arch of T and $Y = C2$.



(c) A is a transfer arch of T and $Y = C7$.



(d) A is thick and not a transfer arch of T, $g \xleftarrow{v_3v_4} C7$ and $Y = C7$.

Figure 9: Case $m_A = 3$.

A is a transfer arch of T:

Then (f, v_2v_3) is a transfer pair of T. Since the exit pair of T satisfies $C2$, we have $f \xleftarrow{v_2v_3} C7$ for both $Y = C2$ and $Y = C7$. If v_1v_2 is not incident to g, (*) contradicts $w(f) \leq 3$. Hence, v_1v_2 is incident to g, which implies $f(B) = g$ (see

Figures 9b and 9c). We do not have $f \xleftarrow{v_1 v_2} C3$, as this contradicts $w(f) \leq 3$. If v_3v_4 is not incident to f, (f, v_1v_2) satisfies all requirements of $C3$ (in particular, v_2v_3 is then an extremal C-edge of both A and f, and v_1v_2 is not an extremal C-edge of a 3-arch by planarity), which contradicts that we do not have $f \xleftarrow{v_1 v_2} C3$. Hence, v_3v_4 is incident to f. Since (f, v_2v_3) is a transfer pair, this implies that v_4 is an extremal C-vertex of f.

Assume that $Y = C2$ (see Figure 9b for the case $m_h = 2$). Since $w(f) \leq 3$ and $\{v_{-1}v_0, v_1v_2\}$ contributes weight at least one to f by (*), we have $g \xleftarrow{v_3 v_4} X$ for some condition X. As f is thick and minor and $m_g \geq 3$, $X \notin \{C1, C2\}$. Since f is incident to v_3v_4 and v_1v_2 is incident to g, $X \notin \{C3, C4, C5, C6\}$, so that $X = C7$. By planarity and the existence of B, (g, v_3v_4) is then a transfer pair and v_3 is an extremal C-vertex of a transfer arch, which contradicts that v_1v_2 is incident to g.

Hence, $Y = C7$. Then, since $v_{-2}v_{-1}$ is not incident to f, either v_{-1} or v_0 is an extremal C-vertex of f. By Lemma 11(vi), v_{-1} is an extremal C-vertex of f. Then C is extendable by Figure 9c.

A is not a transfer arch of T:

Then (f, v_2v_3) is not a transfer pair of T, which implies that v_3v_4 is incident to f and that, if v_1v_2 is incident to g, v_4v_5 is incident to f (as f is thick, g is minor, $m_g \geq 3$ and $g \neq h$).

We show that the weight contribution of $S := \{v_2v_3, v_3v_4, v_4v_5\}$ to f is at least two and, if v_1v_2 is incident to g, at least three. This contradicts $w(f) \leq 3$ because of (*). Let $e := v_iv_{i+1}$ be an edge of S such that e is incident to f and some condition X satisfies $g \xleftarrow{e} X$; we may assume that e exists, as otherwise S satisfies the claim, since v_4v_5 is incident to f if v_1v_2 is incident to g by the result above.

Since e is incident to the minor face f, $X \neq C1$. Since $m_g \geq 3$, $X = C2$ implies $m_g = 3$, so that the definition of $C2$ contradicts that f is thick; hence $X \neq C2$. Assume $X = C3$. By planarity and $m_B \geq 3$, $e \neq v_2v_3$. Since e and the edge $v_{i-1}v_i$ are incident to f, Condition $C3$ implies that $v_{i-2}v_{i-1}$ is not incident to g. As v_2v_3 is incident to g, $C3$ implies $e = v_3v_4$ such that v_4v_5 is incident to a minor face $p \notin \{f, g\}$; in particular, p is neither major nor f. Since (f, v_2v_3) is not a transfer pair of T, then v_1v_2 is neither incident to a major face nor to g, as all other prerequisites for this transfer pair are met. Since H has no minor 1-face, v_1v_2 is incident to h, so that $Y = C7$ and either v_{-1} or v_0 is an extremal C-vertex of f. By Lemma 11(vi), v_{-1} is an extremal C-vertex of f. Then C is extendable by the replacement shown in Figure 9c (disregarding the different distribution of faces).

Assume that $X \in \{C4, C5\}$. Since $m_B \geq 3$ and v_2v_3 is incident to g, we have $e = v_3v_4$ such that v_1v_2 is not incident to g and v_4v_5 is incident to f. Then S contributes weight at least two to f by Lemma 15, as claimed. By the existence of B, Lemma 10, and the fact that v_3v_4 is incident to f and v_2v_3 to g, we have $X \neq C6$.

We conclude that $X = C7$, so that (g, e) is a transfer pair. Let D be the transfer arch that has extremal C-edge e such that $f(D) = g$. Since A is not a transfer arch of T, D is not a transfer arch of T. By Lemmas 11(v) and 12 and $h \overset{b}{\leftarrow} Y$, D is not in the same tunnel as T. In particular, D is not consecutive to A, so that $e \neq v_2v_3$ and D has C-edge v_5v_6. Since (g, e) is a transfer pair and v_2v_3 is incident to g, we have $e = v_3v_4$ such that v_1v_2 is not incident to g and v_5v_6 is not incident to f (see Figure 9d); for the same reason, v_4v_5 is incident to f, as f has no 3-arch that is consecutive to D. Since we excluded all other options for e and X, $\{v_2v_3, v_4v_5\} \subseteq S$ contributes weight at least two to f. This gives the claim, as v_1v_2 is not incident to g.

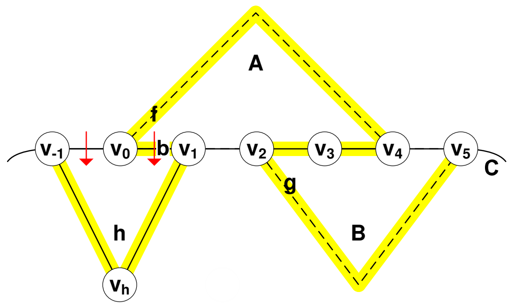
Case $m_A = 4$:

Let g be the (possibly major) v_2v_3-opposite face of f in Figure 10a. Since h is thick, g is thick. As argued at the very beginning of case $m_A = 3$, either v_1 or v_2 is an extremal C-vertex of h, which implies $g \neq h$.

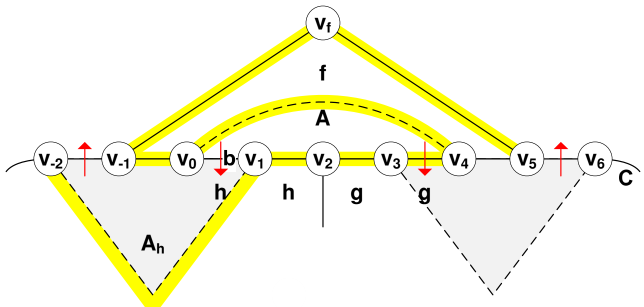
We show that $\{v_1v_2\}$ contributes weight at least one to f and, if v_1v_2 is incident to h, $\{v_{-1}v_0, v_1v_2\}$ contributes weight at least two to f. First, assume to the contrary that the v_1v_2-opposite face of f pulls weight over v_1v_2 by some condition X. Since f is minor, $X \neq C_1$. By planarity, minimality of m_A (which implies that v_2 is not an extremal C-vertex of any arch of A) and $\deg_G(v_2) \geq 3$, $X \notin \{C2, C7\}$. By planarity, $X \neq C_3$. Since v_1 or v_2 is an extremal C-vertex of h, $X \notin \{C4, C5, C6\}$. Hence, the v_1v_2-opposite face of f does not pull weight over v_1v_2. This gives the first claim, so assume that v_1v_2 is incident to h. Then $Y = C7$ and (h, b) is a transfer pair. This implies that $v_{-1}v_0$ is incident to f, as A has no 3-arch with extremal C-vertex v_0 by minimality of m_A. Then v_{-1} is an extremal C-vertex of f and h does not pull weight over $v_{-1}v_0$ using $C3$. By Lemma 10 and $g \neq h$, h does not pull weight over $v_{-1}v_0$ using any other condition. This gives the claim.

Assume to the contrary that g is major. Then $f \overset{v_2v_3}{\longleftarrow} C1$ and v_1v_2 is incident to a minor face. Since H has no minor 1-face, v_1v_2 is incident to h, which contradicts $w(f) \leq 3$ by the claim just proven. Hence, g is thick and minor. If v_3v_4 is not incident to g, g is a thick minor 2-face, as $h \neq g$ and H has no minor 1-face. This contradicts that no opposite face of f pulls weight over a non-extremal C-edge of A using $C2$ or $C7$ (which we proved by minimality of m_A). We conclude that $g \neq h$ is thick and minor and incident to v_3v_4.

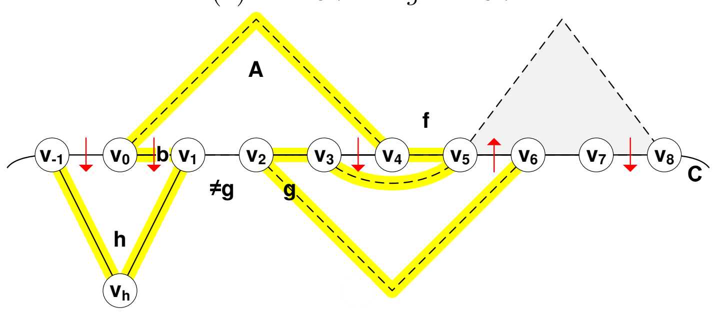
Assume to the contrary that an opposite face of f pulls weight over a C-edge of A different from b using $C2$ or $C7$. Since this is impossible for non-extremal C-edges of A, this edge is v_3v_4 and we have (by $C2$ and the fact that g is incident to v_2v_3) $g \overset{v_3v_4}{\longleftarrow} C7$. Since $g \neq h$, v_5v_6 is an extremal C-edge of a 3-arch of g. By symmetry of A_h and this 3-arch, we may apply the result of the previous paragraph (g is not major) to this 3-arch, which gives that v_1v_2 is not incident to a major face. Since (g, v_3v_4) is a transfer pair, v_1v_2 is then incident to h and thus $Y = C7$. By minimality of m_A, neither v_0 nor v_4 is an extremal C-vertex of a 3-arch of A. Since (h, b) and (g, v_3v_4) are transfer pairs, this implies that f has extremal C-vertices v_{-1} and v_5. Then f is thick and C extendable by Figure 10b. We conclude that b is the only C-edge of A over which an opposite face of f pulls weight using $C2$ or $C7$.



(a) $Y = C2$ and g has a 3-arch B with middle C-edge v_3v_4.



(b) $Y = C7$ and $g \xleftarrow{v_3v_4} C7$.



(c) $Y = C2$, $g \xleftarrow{v_3v_4} C5$ and $v_3v_5 \in E(G)$.

Figure 10: Case $m_A = 4$.

Assume to the contrary that g has a 3-arch B with middle C-edge v_3v_4. Then C is extendable by Figure 10a (and an analogous replacement for $m_h = 3$) if $Y = C2$. If $Y = C7$, we use the case distinction of Lemma 11(iii) to prolong the path replacement to the exit face of the tunnel: C is extendable by the exact path replacement used in the proof of Lemma 11(iii) (for example, the one in Figure 7a) after exchanging its 2-arch with $v_1v_0Av_4v_3v_2Bv_5$ (this adds at most $3 + n_5(G)$ new vertices to C, as A and B add at most one each).

Assume to the contrary that A has a C-edge $e \neq b$ such that the e-opposite face of f pulls weight over e by some condition $X \notin \{C2, C7\}$. Since f is minor, $X \neq C1$. If $X = C3$, we have $e = v_3v_4$ by Lemma 10 and planarity, which contradicts that g has no 3-arch with middle C-edge v_3v_4. Assume $X \in \{C4, C5\}$. By Lemma 10, $h \neq g$ and planarity, we have $e = v_3v_4$ and f is thick. Since A has no 3-arch with extremal

C-vertex v_0, G contains v_1v_3 or v_3v_5 by Lemma 10. In the latter case, C is extendable by Figure 10c if $Y = C2$, and by appending $v_1v_0Av_4v_5v_3v_2v_6$ to the path replacement used in Lemma 11(iii) if $Y = C7$. In the former case, C is extendable by the replacement $v_0Av_4v_3v_1v_2v_6v_5v_bv_7$ for a thick minor 2-face b if $X = C4$, and, similar as before, by prepending $v_0Av_4v_3v_1v_2v_6v_5$ to the path replacement of Lemma 11(iii) if $X = C5$. By Lemma 10, planarity, $\deg_G(v_3) \geq 3$ and the fact that A has no 3-arch with extremal C-vertex v_0, we have $X \neq C6$. Hence, every edge in $\{v_1v_2, v_2v_3, v_3v_4\}$ contributes weight at least one to f.

This gives $w(f) \geq 2$ if f is thin, so let f be thick. If v_1v_2 is incident to h, the contribution of $\{v_{-1}v_0, v_1v_2\}$ to f of weight at least two contradicts $w(f) \leq 3$. If v_1v_2 is incident to a major face, $f \xleftarrow{v_1v_2} C1$ contradicts $w(f) \leq 3$. Hence, v_1v_2 is incident to g and so is v_3v_4, which gives $m_g \geq 4$, as $\{v_1, v_4\}$ is not a 2-separator of G. Assume to the contrary that v_4 is an extremal C-vertex of f. If $Y = C2$, $f \xleftarrow{v_2v_3} C4$ contradicts $w(f) \leq 3$; hence, let $Y = C7$. Then v_0 is not an extremal C-vertex of a 2-arch of h or f by Lemma 11(iii) or (iv), (h, b) is a transfer pair, and v_3v_4 is not the middle C-edge of a 3-arch of g. This implies $f \xleftarrow{v_2v_3} C5$, which contradicts $w(f) \leq 3$. We conclude that v_4v_5 is incident to f and g.

We show that $\{v_4v_5\}$ contributes weight at least one to f, which contradicts $w(f) \leq 3$. Assume to the contrary that g pulls weight over v_4v_5 by some Condition X. As f is minor and $m_g \geq 4$, $X \notin \{C1, C2\}$. Since $m_g \geq 4$, and v_3v_4 is incident to f and v_2v_3 to g, we have $X \neq C3$. By planarity and the facts that v_2v_3 is incident to g and A has no 3-arch with extremal C-vertex v_0, $X \notin \{C4, C5, C6\}$. Hence, $X = C7$, so that (g, v_4v_5) is a transfer pair, which contradicts that v_2v_3 is incident to g.

Case $m_A \geq 5$:

Let $b = v_0v_1$ such that v_0 is an extremal C-vertex of A, let $z := m_A$ and let S be the set of C-edges of A that are different from b. By precisely the same arguments as used at the beginning of Case $m_A = 4$, v_2v_3 is not incident to h, $\{v_1v_2\}$ contributes weight at least one to f and, if v_1v_2 is incident to h, $\{v_{-1}v_0, v_1v_2\}$ contributes weight at least two to f. Since $w(f) \leq 3 < |S|$, S contains an edge that contributes weight zero to f. In the following, we investigate which edges of S may contribute weight zero to f.

Assume that S contains a non-mono edge e that contributes weight zero to f. Since $\{v_1v_2\}$ contributes weight at least one to f, $e \neq v_1v_2$. Let X be the condition by which the e-opposite face p of f pulls weight over e. Since f is minor, $X \neq C1$. Assume $X \in \{C2, C7\}$. Then $e = v_{z-1}v_z$, as no opposite face of f pulls weight over a non-extremal C-edge of A using $C2$ or $C7$. By applying the symmetric version of the statement about the contribution of $\{v_1v_2\}$ above, $\{v_{z-2}v_{z-1}\}$ contributes weight at least one to f and, if $v_{z-2}v_{z-1}$ is incident to p, $\{v_{z-2}v_{z-1}, v_zv_{z+1}\}$ contributes weight at least two to f. If $v_{z-2}v_{z-1}$ is mono, a 3-arch of f pulls weight over $v_{z-2}v_{z-1}$ by $C3$. Since this 3-arch contradicts the minimality of m_A, $v_{z-2}v_{z-1}$ is non-mono. Assume $X \in \{C3, C4, C5, C6\}$. Since e is non-mono, Lemma 14(i) and $e \notin \{b, v_1v_2\}$ imply $e \in \{v_{z-2}v_{z-1}, v_{z-1}v_z\}$. By Lemma 14(ii) and the result for $X \in \{C2, C7\}$, $\{v_{z-2}v_{z-1}, v_{z-1}v_z\}$ contributes weight at least one to f and no edge of $\{v_{z-3}v_{z-2}, v_{z-2}v_{z-1}, v_{z-1}v_z\}$ is mono. We conclude in all cases that

- e is the only non-mono edge of S that contributes weight zero to f,
- $e \in \{v_{z-2}v_{z-1}, v_{z-1}v_z\}$,
- $v_{z-2}v_{z-1}$ and $v_{z-1}v_z$ are non-mono and, if $X \notin \{C2, C7\}$, $v_{z-3}v_{z-2}$ is non-mono, and
- the weight contribution of $\{v_{z-2}v_{z-1}, v_{z-1}v_z, v_zv_{z+1}\}$ to f is at least one and, if $X \in \{C2, C7\}$ and $v_{z-2}v_{z-1}$ is incident to p, at least two.

Assume now that S contains a mono edge e' that contributes weight zero to f. Since $\{v_1v_2\}$ contributes weight at least one to f, $e' \neq v_1v_2$. Let p' be the e'-opposite face of f. Then $p' \overset{e'}{\leftarrow} C3$, so that p' is a minor 3-face by definition of $C3$. Since h is thick, p' is thick. By Lemma 15, each of the two extremal C-edges of p' contributes weight at least one to f; by Lemma 14(ii), every two mono edges in S have distance at least three in C. Since $w(f) \leq 3$, we thus conclude that e' is the only mono edge of S.

We conclude that S contains at most two edges that contribute weight zero to f, namely the non-mono edge e and the mono edge e' above. Since $h \overset{b}{\leftarrow} Y$, $w(f) \geq m_A - 3$. Since $w(f) \leq 3$, this implies $5 \leq m_A \leq 6$ and $w(f) \geq 2$. If f is thin, this gives the claim, so assume that f is thick. We distinguish the following cases.

Subcase $m_A = 5$:

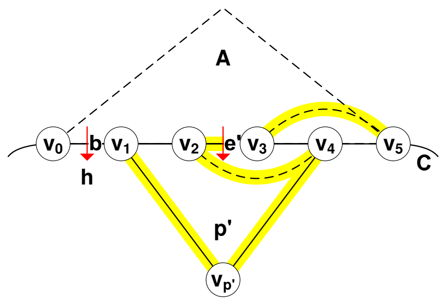
First, assume that e' exists. If $e' = v_2v_3$, Lemma 10 implies $v_2v_5 \notin E(G)$, as p' is a thick minor 3-face. Since $\{v_1, v_4\}$ is not a 2-separator of G and v_0 is not an extremal C-vertex of any arch of A except A itself, G contains v_3v_5 and v_2v_4. Then C is extendable by Figure 11a. Since $e' \neq v_1v_2$, we thus have $e' = v_3v_4$. If the non-mono edge e exists, the results about e above state that $v_{z-2}v_{z-1} = v_3v_4$ is non-mono, which contradicts the choice of e'. Hence, e does not exist, so that $w(f) = 3$. If v_1v_2 is incident to h, the contribution of $\{v_{-1}v_0, v_1v_2\}$ to f contradicts $w(f) \leq 3$. Otherwise, v_1v_2 is incident to a major face, since H has no minor 1-face. This contradicts $w(f) \leq 3$.

Hence, assume that e' does not exist. Then e exists, since S contains an edge that contributes weight zero to f, and we have $w(f) = 3$. In particular, f has no opposite major face and v_1v_2 is not incident to h. Let $X \neq C1$ and p be defined as before in this case and assume $X \in \{C2, C7\}$. By the results about non-mono edges, then v_3v_4 is not incident to p. Hence, v_2v_3 is the middle C-edge of a minor 3-face, which is thick, as h is thick. Then v_2v_3 is mono by $C3$ and Lemma 10 (for $C3$), which contradicts that e' does not exist.

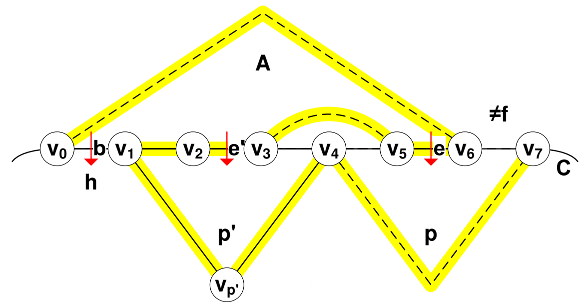
Hence, $X \in \{C3, C4, C5, C6\}$. Since v_1v_2 is neither incident to h nor to a major face nor to a minor thick 2-face, Lemma 15 implies $X = C3$. Then $e = v_{z-1}v_z$, as $e = v_{z-2}v_{z-1}$ contradicts by $C3$ that e is non-mono. By definition of $C3$, then v_2v_3 is not incident to p. Since v_2v_3 is neither incident to h nor to a minor thick 2-face, v_2v_3 is incident to a major face, which is a contradiction.

Subcase $m_A = 6$:

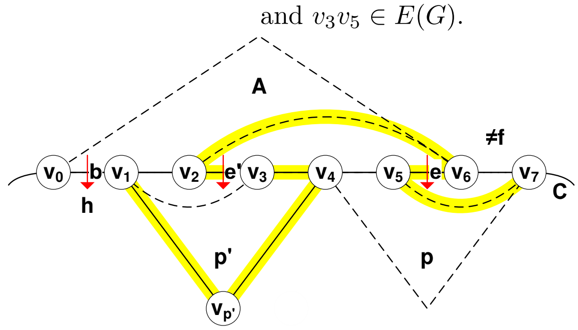
Then $w(f) = 3$, e and e' exist, v_1v_2 is not incident to h, and f has no opposite



(a) $m_A = 5$ and a mono edge e'.



(b) $m_A = 6$, a mono edge e', a non-mono edge e and $v_3v_5 \in E(G)$.



(c) $m_A = 6$, a mono edge e', a non-mono edge e and $v_3v_5 \notin E(G)$.

Figure 11: Case $m_A \geq 5$.

major face (see Figure 11b). Since H has no minor 1-face, $e' = v_2v_3$, so that v_1v_2 is incident to p'. Then $X \notin \{C2, C7\}$, as otherwise $v_{z-2}v_{z-1}$ is either incident to a major face or to p, which contradicts $w(f) = 3$ due to the contribution of $\{v_{z-2}v_{z-1}, v_{z-1}v_z, v_zv_{z+1}\}$.
Hence, $X \in \{C3, C4, C5, C6\}$. By Lemma 15, $X \in \{C3, C6\}$. By Lemma 10 for $C3$ (and p'), we have $v_1v_5 \notin E(G)$ and $v_2v_5 \notin E(G)$; by minimality of m_A, we still have $v_0v_5 \notin E(G)$. If $X = C6$, Lemma 10 (for p) thus contradicts $\deg_G(v_5) \geq 3$. Hence, $X = C3$. If $v_3v_5 \in E(G)$, C is extendable by Figure 11b. Otherwise, $v_3v_5 \notin E(G)$, so that $\deg_G(v_5) \geq 3$ implies $v_5v_7 \in E(G)$. By Lemma 10 for $C3$ (and p), $v_3v_6 \notin E(G)$. By minimality of m_A, no arch of A except A itself has C-vertex v_0. Since $\{v_1, v_4\}$ is not a 2-separator of G, we have $v_2v_6 \in E(G)$. Then C is extendable by Figure 11c.

This completes the proof. □

In particular, we may choose A in Lemma 16 as the proper arch of f. This implies the following helpful corollary.

Corollary 17. *Every minor face f of H that has a C-edge over which an opposite face of f pulls weight by $C2$ or $C7$ satisfies $w(f) \geq 2$ and, if f is thick, $w(f) \geq 4$.*

We now show that Inequalities (1) and (2) hold, which proves the Isolation Lemma.

Lemma 18. *Let f be a face of H. Then*

- $w(f) \geq 0$ *if f is major,*
- $w(f) \geq 2$ *if f is minor and thin, and*
- $w(f) \geq 4$ *if f is minor and thick.*

Proof. By Lemma 13, any opposite face of f pulls over any C-edge of f weight at most one. Since f is initially charged with weight m_f, the weight $w(f)$ after applying our discharging rule satisfies $w(f) \geq 0$. In the remaining part of the proof, let f be minor. Assume that f has a C-edge e' such that the e'-opposite face of f pulls weight over e' by Condition C2 or C7. Then the claim follows by Corollary 17. We therefore assume throughout the proof that no opposite face of f pulls weight over a C-edge of f by Condition $C2$ or $C7$.

Let $m_f = 2$. If f is thick, Condition $C2$ and Lemma 13 imply $w(f) = 4$. If f is thin, assume to the contrary that $w(f) \leq 1$. Then f has a C-edge e such that $g \overset{e}{\leftarrow} X$ for the e-opposite face g of f. By our assumptions, $X \in \{C3, C4, C5, C6\}$. Since $m_f = 2$, $X \notin \{C3, C4, C5\}$. Thus, $X = C6$, which contradicts Lemma 10.

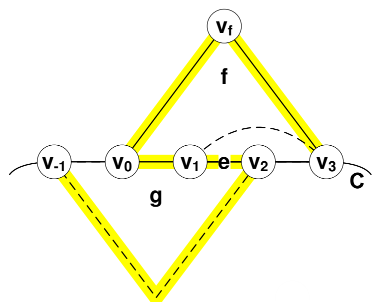
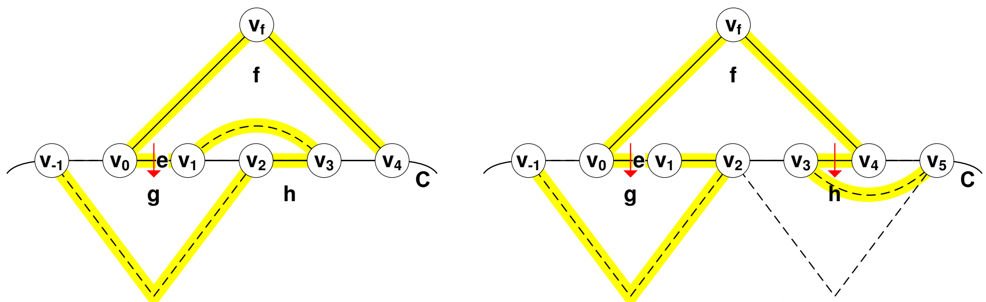


Figure 12: $m_f = 3$ when e is an extremal C-edge of a 3-arch.

Let $m_f = 3$. Assume to the contrary that f has a C-edge b such that $h \overset{b}{\leftarrow} X$ for the b-opposite face h of f. Then every $X \in \{C3, C4, C5, C6\}$ contradicts Lemma 10. Hence, $w(f) \geq 3$. If f is thin, this gives the claim, so let f be thick. If f has an opposite major face, $C1$ implies the claim $w(f) \geq 4$, so assume otherwise. Let e be the middle C-edge of f and let g be the minor e-opposite face of f. Then e is not an extremal C-edge of a 3-arch, as otherwise C is extendable by Figure 12. If $m_g \geq 3$, we therefore have $f \overset{e}{\leftarrow} C3$, which gives $w(f) \geq 4$. Otherwise, $m_g = 2$, since H has no minor 1-face. By symmetry, say that g has extremal C-vertices v_0 and v_2; then $\deg_G(v_1) \geq 3$ implies $v_1v_3 \in E(G)$, and Lemma 10 (for $C2$) implies that g is thin. Then $f \overset{v_2v_3}{\longleftarrow} C2$, which gives $w(f) \geq 4$.

Let $m_f = 4$. We may assume that f has a C-edge e such that $g \overset{e}{\leftarrow} X$ for the e-opposite face g of f, as otherwise $w(f) \geq m_f = 4$. If $e = v_1v_2$ (or $e = v_2v_3$ by symmetry), Lemma 10 implies $X \notin \{C3, C4, C5\}$ because $m_f = 4$ and we have $X \neq C6$ by definition of $C6$. We conclude that $\{v_1v_2, v_2v_3\}$ contributes weight at least two to f and that e is an extremal C-edge of f, say $e = v_0v_1$ by symmetry. If f is thin, this gives the claim $w(f) \geq 2$, so let f be thick. Since $e = v_0v_1$, $X \notin \{C4, C5\}$, as $v_{-1}v_0$ is not incident to f. Hence, $X \in \{C3, C6\}$, which implies in both cases that v_2v_3 is incident to a face $h \notin \{f, g\}$. By



(a) $e = v_0v_1$, $X = C3$ and $v_1v_3 \in E(G)$.

(b) $e = v_0v_1$ and v_2v_3 is an extremal C-edge of a 3-arch of h.

Figure 13: $m_f = 4$.

Lemma 10 (for $C3$ and $C6$), $v_0v_3 \notin E(G)$. In addition, $v_1v_3 \notin E(G)$, as otherwise $X = C3$ implies that C is extendable by Figure 13a and $X = C6$ contradicts Lemma 10 (for $C6$).

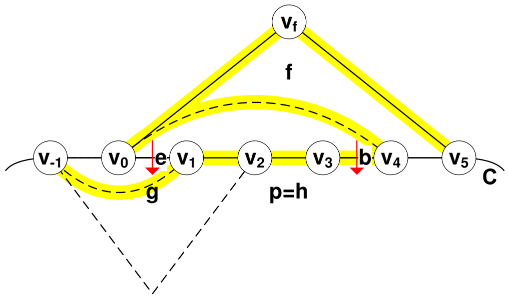
If v_2v_3 is an extremal C-edge of a 3-arch, $v_0v_3 \notin E(G)$ and $v_1v_3 \notin E(G)$ imply $v_3v_5 \in E(G)$; then C is extendable by Figure 13b, since f is thick. Hence, v_2v_3 is not an extremal C-edge of a 3-arch. If an opposite face of f pulls weight over v_3v_4 by some Condition Y, $Y \in \{C3, C6\}$ by the same argument as for X; since $v_0v_3 \notin E(G)$ and $v_1v_3 \notin E(G)$, Lemma 10 (for $C6$) implies then $Y = C3$, so that v_2v_3 is an extremal C-edge of a 3-arch, which is impossible. Hence, no opposite face of f pulls weight over v_3v_4.

If h is major, we thus have $w(f) \geq 4$ by $C1$, so let h be minor. Since H has no minor 1-face and v_2v_3 contributes weight at least one to f, $m_h \geq 3$. If $v_1v_4 \in E(G)$, we have $f \xleftarrow{v_2v_3} C3$, as v_2v_3 is not an extremal C-edge of a 3-arch, h is minor, $m_h \geq 3$, v_4v_5 is not incident to f and v_1v_2 is not incident to h. Since this gives the claim, let $v_1v_4 \notin E(G)$. If $X = C3$, then $f \xleftarrow{v_2v_3} C6$, which gives the claim. If otherwise $X = C6$, then $\deg_G(v_1) \geq 3$, $v_1v_3 \notin E(G)$ and Lemma 10 (for $C6$) imply $v_1v_4 \in E(G)$, which is a contradiction.

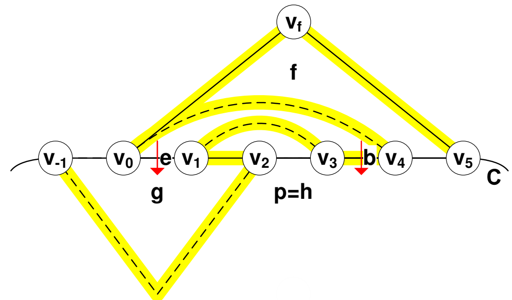
Let $m_f \geq 5$. By Lemma 14(ii), f has at most $\lfloor \frac{m_f+2}{3} \rfloor$ C-edges over which an opposite face of f pulls weight. Hence, $w(f) \geq m_f - \lfloor \frac{m_f+2}{3} \rfloor = \lceil \frac{2}{3}(m_f - 1) \rceil$. This gives the claim if f is thin or $m_f \geq 6$, so let f be thick, $m_f = 5$ and $w(f) = 3$.

Then f has exactly two C-edges e and b such that $g \xleftarrow{e} X$ and $h \xleftarrow{b} Y$ for opposite faces g and h of f, and no major opposite face. By Lemma 14(i) and (ii), e or b is an extremal C-edge of f, say $e = v_0v_1$ and $b \in \{v_3v_4, v_4v_5\}$ by symmetry. Since $v_{-1}v_0$ is not incident to f, $X \in \{C3, C6\}$, which implies in both cases that v_2v_3 is incident to a face $p \notin \{f, g\}$. Assume to the contrary that $b = v_4v_5$. Since v_5v_6 is not incident to f, $Y \in \{C3, C6\}$, which implies in both cases that $p \notin \{f, g, h\}$. Since H has no minor 1-face, p is major, which contradicts our assumption. We conclude that $b = v_3v_4$, so that $Y \in \{C3, C4, C5\}$ by definition of $C6$.

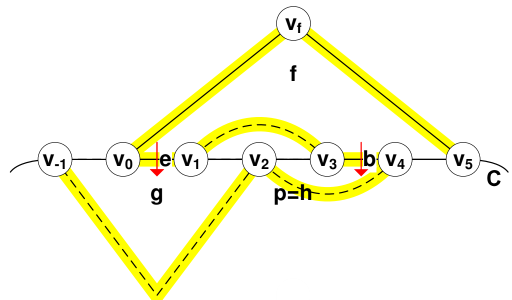
By Lemma 10 for $X \in \{C3, C6\}$ and $Y \in \{C3, C4, C5\}$, G does not contain any edge of $\{v_0v_3, v_1v_4, v_1v_5, v_2v_5\}$. Since $\deg_G(v_1) \geq 3$, $v_{-1}v_1 \in E(G)$ or $v_1v_3 \in E(G)$. As $X = C6$ implies $v_{-1}v_1 \notin E(G)$ and $v_1v_3 \notin E(G)$ by Lemma 10, $X = C3$. If $v_0v_4 \in E(G)$, C is extendable by Figure 14a when $v_{-1}v_1 \in E(G)$ and by Figure 14b otherwise. Hence, assume $v_0v_4 \notin E(G)$. By Lemma 10 for $Y \in \{C3, C4, C5\}$ and the fact that $\{v_2, v_5\}$ is not a



(a) $v_0v_4 \in E(G)$ and $v_{-1}v_1 \in E(G)$.



(b) $v_0v_4 \in E(G)$ and $v_1v_3 \in E(G)$.



(c) $v_0v_4 \notin E(G)$.

Figure 14: $m_f = 5$ when $b = v_3v_4$ and $X = C3$.

2-separator of G, a vertex of $\{v_0, v_1\}$ is adjacent to a vertex of $\{v_3, v_4\}$. By the previous results, $v_1v_3 \in E(G)$ and $v_2v_4 \in E(G)$. Then C is extendable by Figure 14c. □

4 Algorithms

We conclude this paper with algorithmic versions of the Isolation Lemma and of Corollary 4.

Theorem 19. *Given an isolating cycle C of length $c < \min\{\lfloor \frac{2}{3}(n+4) \rfloor, n\}$ of a polyhedral graph G on n vertices, a larger isolating cycle C' of G that satisfies $V(C) \subset V(C')$ and $|V(C')| \leq |V(C)| + 3 + n_5(G)$ can be computed in time $O(n)$.*

Proof. We compute the graph H and identify all minor faces of H, their C-edges, the information whether they are thick or thin, and all tunnels in linear time $O(n)$. If H has a thick minor 1-face f, extending C by the arch of f in constant time gives the claim. Otherwise, for every of the $2c = O(n)$ relevant face-edge pairs (g, e), there is only a constant number of configurations in which C is extended, according to the proof of the Isolation Lemma. We identify these cases and extend C in total time $O(n)$. □

Theorem 20. *Given an essentially 4-connected planar graph G on n vertices, an isolating Tutte cycle of G of length at least $\min\{\lfloor \frac{2}{3}(n+4) \rfloor, n\}$ can be computed in time $O(n^2)$.*

Proof. If $n \leq 10$, G is Hamiltonian, as described in the proof of Corollary 4, so that we may compute even a Hamiltonian isolating cycle of G in constant time. Hence, assume

$n \geq 11$. For computing a first isolating Tutte cycle C of G, we choose the start and end-edges of a Tutte path carefully as described in [4, Lemma 4(i)]. This can be done in time $O(n^2)$ by using the algorithm from [13] (we note that the faster algorithm by Biedl and Kindermann [2] cannot be used here, as it does not allow to prescribe the start and end-edges). Then applying Theorem 19 iteratively to C gives the claim. □

Acknowledgments. The authors want to thank the anonymous reviewers for their detailed and helpful comments and the second author wants to thank Jochen Harant for introducing him to this interesting topic.

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