On the Circumference of Essentially 4-connected Planar Graphs

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Abstract

A planar graph is *essentially* 4*-connected* if it is 3-connected and every of its 3-separators is the neighborhood of a single vertex. Jackson and Wormald proved that every essentially 4-connected planar graph G on n vertices contains a cycle of length at least $\frac{2n+4}{5}$, and this result has recently been improved multiple times.

In this paper, we prove that every essentially 4-connected planar graph G on n vertices contains a cycle of length at least $\frac{5}{8}(n+2)$. This improves the previously best-known lower bound $\frac{3}{5}(n+2)$.

1 Introduction

The *circumference* $\text{circ}(G)$ of a graph G is the length of a longest cycle of G. Originally being the subject of Hamiltonicity studies, essentially 4-connected planar graphs and their circumference have been thoroughly investigated throughout literature. Jackson and Wormald [5] proved that $\text{circ}(G) \geq \frac{2n+4}{5}$ for every essentially 4-connected planar graph G on n vertices. An upper bound is given by an infinite family of essentially 4-connected planar graphs G such that $\text{circ}(G) = \frac{2}{3}(n+4)$ [2]. Fabrici, Harant and Jendroľ [2] improved recently the lower bound to $\text{circ}(G) \geq \frac{1}{2}(n+4)$; this result in turn was strengthened to $\text{circ}(G) \geq \frac{3}{5}(n+2)$ in [3]. It remained an open problem whether every essentially 4-connected planar graph G on n vertices satisfies $\text{circ}(G) > \frac{3}{5}(n+2)$.

In this paper, we present the following result.

Theorem 1. *Every essentially 4-connected planar graph G on n vertices contains a cycle of length at least $\frac{5}{8}(n+2)$. If $n \geq 16$, $\text{circ}(G) \geq \frac{5}{8}(n+4)$.*

This result encompasses most of the results known for the circumference of essentially 4-connected planar graphs (some of which can be found in [2, 4, 8]). In particular, it improves the bound $\text{circ}(G) \geq \frac{13}{21}(n+4)$ that has been given in [2] for the special case that G is maximal planar for sufficiently large n (in fact, for every $n \geq 16$, as explained in Section 4).

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2 Preliminaries

Throughout this paper, all graphs are simple, undirected and finite. For a vertex x of a graph G, denote by $\deg_G(x)$ the degree of x in G. For a vertex subset $A \subseteq V$, let the *neighborhood* $N_G(A)$ of A consist of all vertices in $V - A$ that are adjacent to some vertex of A. For vertices $v_1, v_2, \dots, v_i$ of a graph G, let $(v_1, v_2, \dots, v_i)$ be the path of G that visits the vertices in the given order. We omit subscripts if the graph G is clear from the context.

A *separator* S of a graph G is a subset of V such that $G - S$ is disconnected; S is a *k-separator* if $|S| = k$. A separator S is *trivial* if at least one component of $G - S$ is a single vertex, and *non-trivial* otherwise. Let a graph G be *essentially 4-connected* if G is 3-connected and every 3-separator of G is trivial. It is well-known that, for every 3-separator S of a 3-connected planar graph G, $G - S$ has exactly two components.

A cycle C of a graph G is *isolating* (sometimes also called *outer-independent*) if every component of $G - V(C)$ is a single vertex that has degree three in G. An edge xy of a cycle C of G is *extendable* if x and y have a common neighbor in $G - V(C)$. For example, Figure 2 depicts (a part of) an isolating cycle C for which the edge yz becomes extendable after contracting the edge zu. According to Whitney [7], every 3-connected planar graph has a unique embedding into the plane (up to flipping and the choice of the outer face). Hence, we assume in the following that the embeddings of such graphs are fixed.

3 Proof of Theorem 1

Let G be an essentially 4-connected plane graph. It is well-known that every 3-connected plane graph on at most 10 vertices is Hamiltonian [1]; thus, for $4 \leq n \leq 10$, this implies $\text{circ}(G) = n \geq \frac{5}{8}(n+2)$. Since these graphs contain in particular the essentially 4-connected plane graphs on at most 10 vertices, we assume $n \geq 11$ from now on. For $n \geq 11$, it was shown in [2, Lemma 4(ii)] that G contains an isolating cycle of length at least 8. Let C be a longest such isolating cycle of length $c := |E(C)| \geq 8$. We will show that $c \geq \frac{5}{8}(n+2)$, so that C is a cycle of the desired length.

Clearly, C contains no extendable edge xy, as otherwise one could find a longer such cycle by replacing xy in C with the path (x, v, y), where $v \notin V(C)$ is a common neighbor of x and y. Let V^- be the subset of vertices of V that are contained in the open set of $\mathbb{R}^2 - C$ that is bounded (hence, strictly inside C), and let $V^+ := V - V(C) - V^-$. We assume that $|V^-| \geq 1 \leq |V^+|$, since otherwise we are done, as then $c \geq \frac{2}{3}(n+2)$ is implied by [2, Lemma 5]. Let H be the plane graph obtained from G by deleting all chords of C (i. e., all edges $xy \in E - E(C)$ satisfying $x, y \in V(C)$) and let $H^- := H - V^+$ and $H^+ := H - V^-$. A face of H is called *minor* if it is incident to exactly one vertex of $V^- \cup V^+$, and *major* otherwise. Let M^- and M^+ be the sets of minor faces in H^- and H^+, respectively. For example, in Figure 2, we have $a \in V^-$, $b \in V^+$, $f \in M^-$ and $f' \in M^+$.

Note that a face f of H is incident to no vertex of $V^- \cup V^+$ if and only if it is bounded by C (i.e., if f is either the region inside or outside C). Since we assumed $|V^-| \geq 1 \leq |V^+|$, our definition of minor faces coincides with the one of [3], so that we can use the following inequality.

Lemma 2 ([3], Inequality (i))**.** $|M^- \cup M^+| \geq |V^- \cup V^+| + 2$.

In H, an edge e of C is incident with exactly two faces f and f' of H. In this case we say f' is *opposite* to f with respect to e. A face f of H is called *j-face* if it is incident with exactly j edges of C; the edges of C that are incident with f are called *C-edges* of f. Since C does not contain an extendable edge, we have $j \geq 2$ for every minor j-face of H. For two faces f and f' of H, let $m_{f,f'}$ be the number of common C-edges of f and f'.

If we can prove

$$2c \geq \frac{10}{3}|M^- \cup M^+|, \tag{1}$$

then Theorem 1 follows directly from the inequality $|M^- \cup M^+| \geq n - c + 2$ of Lemma 2. We charge every j-face of H with weight j (and thus have a total charge of weight $2c$) and discharge these weights in H by applying the following set of rules exactly once. In order to prove Inequality (1), we will aim to prove that every minor face of H has weight at least $10/3$ after the discharging.

Rule R1: Every major face f of H sends weight $m_{f,f'}$ to every minor face f' opposite to f.

Rule R2: Every minor face f of H sends weight $\frac{2}{3}m_{f,f'}$ to every minor 2-face f' opposite to f.

Rule R3: Every minor face f of H sends weight 1 to every minor 3-face f' that is opposite to f with respect to the middle C-edge of f'.

Rule R4: Let f_1 be a minor 4-face that has an opposite minor j-face f satisfying $j \geq 4$ and $m_{f_1,f} = 2$, as well as an opposite minor 2- or 3-face f_2 satisfying $m_{f_1,f_2} = 2$. Then f sends weight $2/3$ to f_1.

Rule R5: Let f_1 be a minor 5-face that has an opposite minor j-face f satisfying $j \geq 4$ and $m_{f_1,f} = 2$, as well as two opposite minor 2-faces. Then f sends weight $1/3$ to f_1.

For example, in Figure 2, both faces f and f' would send weight $2/3$ to each other according to Rule R2, which effectively cancels the exchange of weights. Rules R2 and R3 may be seen as a refinement of the two rules given in [3]; for that reason, some of the early cases about minor 2- and 3-faces in the following case distinction will be similar as in [3].

Let w denote the weight function on the set $F(H)$ of faces of H after Rules R1–R5 have been applied. Clearly, $\sum_{f \in F(H)} w(f) = 2c$ still holds. In order to prove that the weight $w(f)$ of every minor face f of H is at least $10/3$ and no major face has negative weight, we distinguish several cases. For most of them, we construct a cycle $\overline{C}$ that is obtained from C by replacing a subpath of C with another path. In such cases, $\overline{C}$ will be an isolating (which is easy to verify due to $V(C) \subseteq V(\overline{C})$) cycle of G that is longer than C (we say C is *extended*); this contradicts the choice of C and therefore shows that the considered case cannot occur. Note that the vertices of C that are depicted in the following figures are pairwise non-identical, because $c \geq 8$; in the rare figures that show more than 8 vertices of C, C has always at least the number of vertices shown.

Let $f \in F(H)$.

Case 1: *f is a major j-face for any j.*

Initially, f is charged with weight j. By Rule R1, f sends for every of its C-edges weight at most 1 to an opposite face. We conclude $w(f) \geq 0$.

Case 2: *f is a minor 2-face* (see Figure 1).

Let xy and yz be the C-edges of f and let a be the vertex of $V - V(C)$ that is incident with f. The face f is initially charged with weight 2 and gains weight at least $4/3$ by R1 and R2. If f does not send any weight to other faces, this gives $w(f) \geq 10/3$, so assume that f sends weight to some face $f' \neq f$.

According to R1–R5, f' is opposite to f and either a minor 2-face or a minor 3-face of H. Without loss of generality, let f' be opposite to f with respect to the edge yz. We distinguish the following subcases.

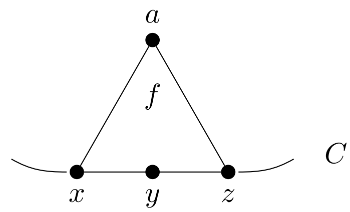

Figure 1: Case 2

Case 2a: *f' is a minor 2-face and xy is a C-edge of f'.*

Then $\{x, z\}$ is the neighborhood of y in G, which contradicts the 3-connectivity of G.

Case 2b: *f' is a minor 2-face and xy is not a C-edge of f'* (see Figure 2).

Then a longer isolating cycle $\overline{C}$ is obtained from C by replacing the path (x, y, z, u) with the path (x, a, z, y, b, u) (see Figure 2), which contradicts the choice of C.

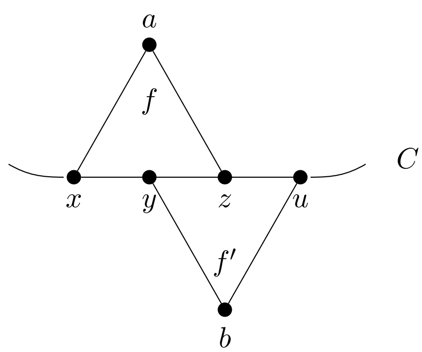

Figure 2: Case 2b

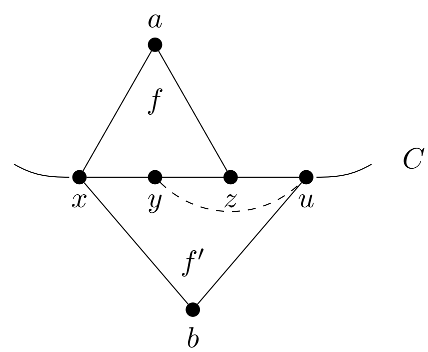

Figure 3: Case 2c

Case 2c: *f' is a minor 3-face* (see Figure 3).

Since we assumed that f sends weight to f', one C-edge of f, say without loss of generality yz, is the middle C-edge of f', according to R3. The edge yu (see Figure 3) exists in G (but not in H, as H does not contain chords of C), because otherwise $d_G(y) = 2$, which contradicts that G is 3-connected. Then $\overline{C}$ is obtained from C by replacing the path (x, y, z, u) with the path (x, a, z, y, u).

Case 3: *f is a minor 3-face* (see Figure 4).

Then f is initially charged with weight 3 and gains weight at least 1 by R1 and R3. If f sends weight at most $2/3$ to other faces, this gives $w(f) \geq 10/3$, so assume that f sends weight more than $2/3$. Since all weights are multiples of $1/3$, f has to send weight at least $3/3$. In particular, this implies that Rule R2 or R3 applies on f.

Let f_1, f_2 and f_3 be the (possibly identical) opposite faces of f with respect to the C-edges vx, xy, yz of f (see Figure 4). Then f_2 is not a minor 2-face for the same reason as in Case 2c. We distinguish the following subcases.

Case 3a: *Neither f_1 nor f_3 is a minor 3-face* (see Figure 5).

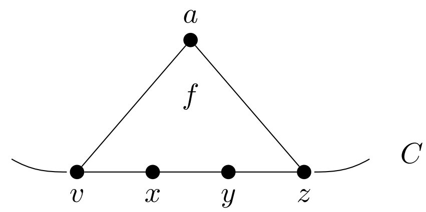

Figure 4: Case 3

Then f_2 is neither a minor 2-face nor a minor 3-face, and f_1 and f_3 are minor 2-faces, as otherwise by R1–R5 f would not send a total weight of more than $2/3$ to its opposite faces. Moreover, $b \neq d$ (in the notation of Figure 5), since xy is not extendable. Then $\overline{C}$ is obtained from C by replacing the path (w, v, x, y, z, u) with the path $(w, b, x, v, a, z, y, d, u)$.

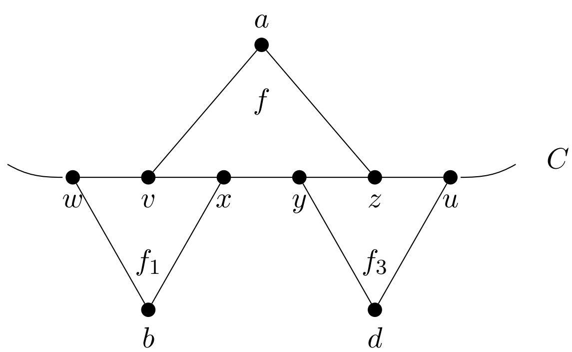

Figure 5: Case 3a

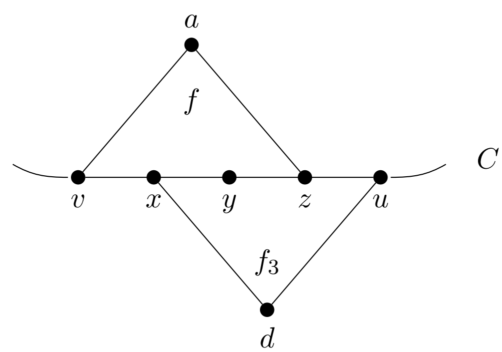

Figure 6: Case 3b

Case 3b: *f_1 or f_3 is a minor 3-face* (see Figure 6).

The face f_2 is not a minor 3-face with middle C-edge xy, as otherwise $\{v, z\}$ would be a 2-separator of G. Hence, $f_1 \neq f_3$. Since f sends a total weight of more than $2/3$ to its opposite faces, at least one of f_1 and f_3 is a minor 3-face that has its middle C-edge in $\{vx, yz\}$ by R3, say without loss of generality that the middle C-edge of f_3 is yz. Then $\overline{C}$ is obtained from C by replacing the path (v, x, y, z, u) with the path (v, a, z, y, x, d, u).

Case 4: *f is a minor 4-face* (see Figure 7).

Then f is initially charged with weight 4. If f looses a total net weight of at most $2/3$, then $w(f) \geq 10/3$, so assume that weight at least $3/3$ is sent to opposite faces. We have to show that this is impossible by considering Rules R2–R5.

Assume first that f has an opposite minor 2-face f'. We distinguish the following subcases.

Case 4a: *f' has C-edges wx and xy* (see Figure 8).

Then vx or xz is an edge of G and C can be extended by detouring C through one of these edges and d, which contradicts the choice of C.

Case 4b: *Every opposite minor 2-face of f has exactly one C-edge of f* (see Figure 9).

In particular, $m_{f,f'} = 1$. Without loss of generality, let f' have the C-edge yz. Then f sends weight $2/3$ to f' by R2, and R1 does not decrease the weight of f. Moreover,

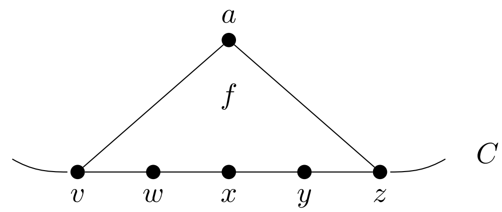

Figure 7: Case 4

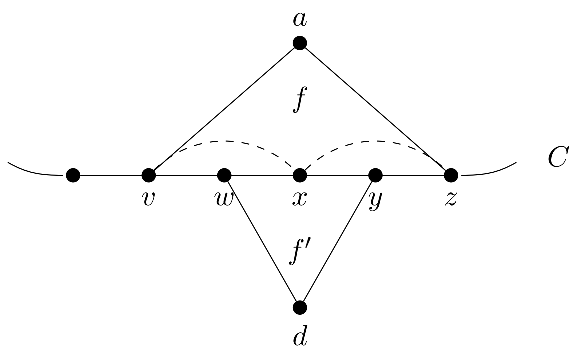

Figure 8: Case 4a

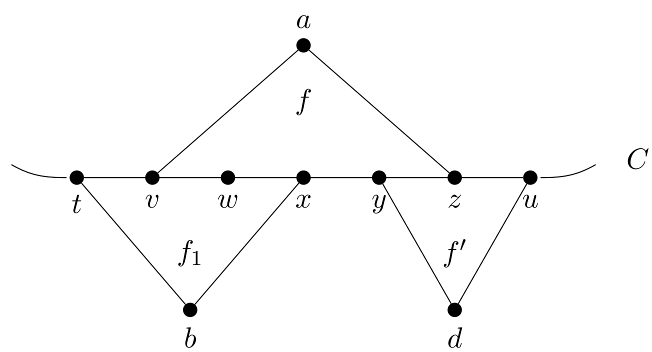

Figure 9: Case 4b

if f sends weight to another face with the Rules R4 or R5, then xy is a C-edge of a major face (since C does not contain any extendable edge) and f gains weight 1 from this major face, so that $w(f) \geq 4 - 2/3 + 1 - 2/3 = 11/3$, which contradicts $w(f) < 10/3$. Therefore, f has by R2 and R3 an opposite minor 2- or 3-face $f_1 \neq f'$. If f_1 is a minor 2-face, $m_{f,f_1} = 1$, so that f_1 has the C-edge vw. Then neither wx nor xy is a C-edge of a minor face opposite to f, as such a minor face would be a 2-face with C-edges wx and xy (see Case 4a). Thus, f gains weight 2 from the major face(s) with C-edges wx and xy, which contradicts $w(f) < 10/3$.

Hence, f_1 is a minor 3-face. Since $w(f) < 10/3$, the middle C-edge of f_1 is either vw or wx. If it is vw, $\overline{C}$ can be obtained from C by replacing the path (t, v, w, x, y, z, u) with $(t, b, x, w, v, a, z, y, d, u)$ (see Figure 9), as we have $b \neq d$, since otherwise C would contain the extendable edge xy. Hence, let the middle C-edge of f_1 be wx. Then $wz \notin E(G)$, as otherwise C could be extended by replacing the path (v, w, x, y, z) with (v, b, y, x, w, z). Since $\{v, y\}$ is not a 2-separator of the 3-connected graph G, this implies $xz \in E(G)$. Then $\overline{C}$ can be obtained from C by replacing the path (x, y, z, u) with (x, z, y, d, u), which contradicts the choice of C.

From Cases 4a+b, we conclude that f' has either the C-edges vw and wx or the C-edges xy and yz, say without loss of generality the latter.

Case 4c: *f' has C-edges xy and yz, and f has an opposite major face* (see Figure 10).

Then $wy \notin E(G)$, as otherwise C can be extended by detouring through f'. Hence, $vy \in E(G)$, as otherwise $\deg_G(y) = 2$. Since f has an opposite major face and wx is not an extendable edge of C, wx is a C-edge of such an opposite major face f''. Then f gains weight 1 from f'' by R1 and sends by R2 weight $2/3$ to a minor opposite 2-face with C-edge vw in order to satisfy the assumption $w(f) < 10/3$ (see Figure 10 and note that R4 and R5 do not apply here). But this is impossible, as then C can be extended by replacing the path (t, v, w, x, y, z) with (t, b, w, v, y, x, d, z), since $b \neq d$.

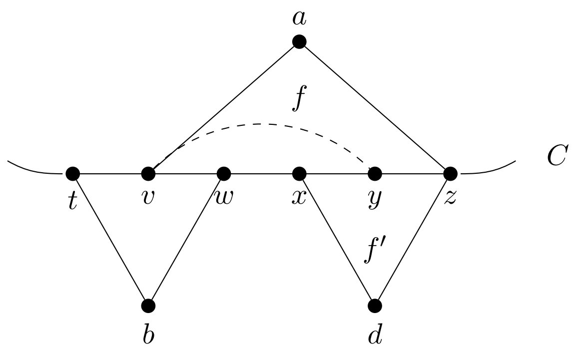

Figure 10: Case 4c

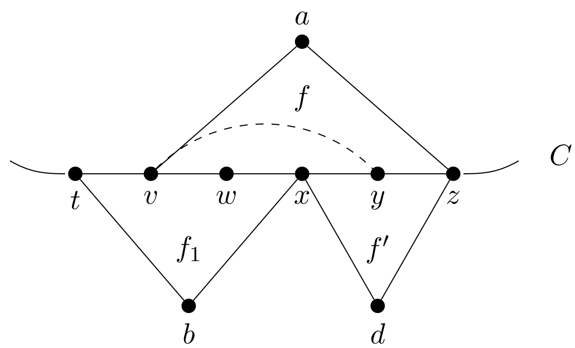

Figure 11: Case 4d

Case 4d: *f' has C-edges xy and yz, and wx is a C-edge of a minor 2- or 3-face f_1* (see Figure 11).

As in Case 4c, $wy \notin E(G)$ and $vy \in E(G)$. Hence, f_1 is a minor 3-face, as otherwise $\deg_G(w) = 2$. Then $\overline{C}$ is obtained from C by replacing the path (t, v, w, x, y, z) with (t, b, x, w, v, y, z) (note that $b = d$ is possible).

Case 4e: *f' has C-edges xy and yz, and wx is a C-edge of a minor j-face f_1 with $j \geq 4$* (see Figure 12).

Then f gains weight $2/3$ from f_1 by R4 and sends weight $4/3$ to f'. Hence, we get the contradiction $w(f) = 10/3$, unless f sends weight $2/3$ to f_1 by R4 or $1/3$ to f_1 by R5. In that case, $j = 4$ or $j = 5$ and there are only minor 2-faces opposite to f_1. As argued in Case 4c, $wy \notin E(G)$ and $vy \in E(G)$. Moreover, uw (and su in case of $j = 5$; see Figure 12) are not edges of G, as otherwise C can be extended by detouring through g. Hence, $ux \in E(G)$, as otherwise $\deg_G(u) = 2$, which is a contradiction. This implies $\deg_G(w) = 2$, which is a contradiction.

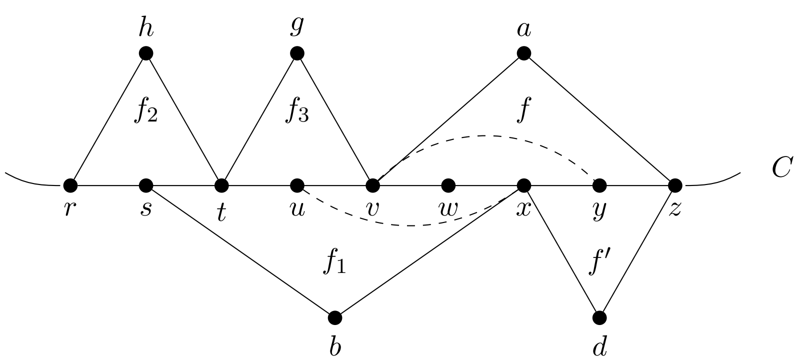

Figure 12: Case 4e

From Cases 4a–e, we conclude that f has no opposite minor 2-face. Then $w(f) < 10/3$ and R1–R5 imply that f has an opposite minor 3-face that has a C-edge of f as middle C-edge (due to R3), or an opposite minor 4-face f' with $m_{f,f'} = 2$ that has an opposite minor 2- or 3-face f_2 with $m_{f',f_2} = 2$ (due to R4); note that we still contradict $w(f) < {}^{10}/_3$ when f has two opposite minor 5-faces, to each of which f sends weight ${}^1/_3$ by R5. We therefore distinguish these remaining subcases.

Case 4f: *f has an opposite minor 3-face f' with middle C-edge wx or xy* (see Figure 13).

Without loss of generality, let xy be the middle C-edge of f'. Then $vy \notin E(G)$, as otherwise C can be extended by replacing the path (v, w, x, y, z) with (v, y, x, w, d, z). This implies $wy \in E(G)$, as otherwise $\deg_G(y) = 2$. Since $\{w, z\}$ is no 2-separator of G, $vx \in E(G)$. Then C can be extended by replacing the path (v, w, x, y, z) with (v, x, y, w, d, z).

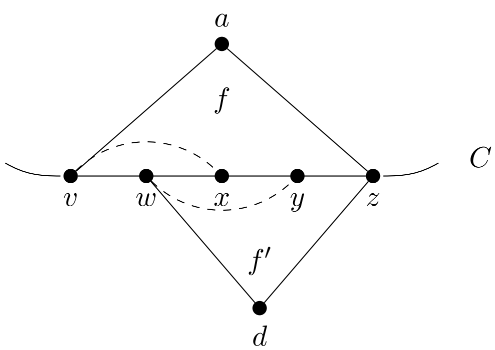

Figure 13: Case 4f

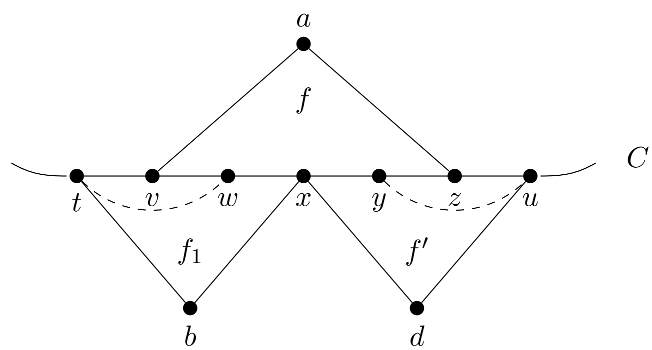

Figure 14: Case 4g

Case 4g: *f has an opposite minor 3-face f' with middle C-edge vw or yz, but no opposite 4-face* (see Figure 14).

Without loss of generality, let yz be the middle C-edge of f'. Let f_1 be the face opposite to f that has C-edge wx. Then f_1 is not major, as otherwise $w(f) = 4 - 1 + 1 > {}^{10}/_3$, since f has no opposite minor 2-faces. For the same reason, f_1 is a minor j-face satisfying $j \geq 3$. If $j \geq 5$, f_1 sends weight ${}^2/_3$ to f due to R4, which contradicts $w(f) < {}^{10}/_3$, as f sends weight at most ${}^1/_3$ to f_1 due to R5 (exactly ${}^1/_3$ only if $j = 5$ and f_1 has two opposite 2-faces).

Since $j \neq 4$ by assumption, f_1 is a minor 3-face (see Figure 14). Then $wy \notin E(G)$, as otherwise $\overline{C}$ is obtained from C by replacing the path (v, w, x, y, z, u) with (v, a, z, y, w, x, d, u), and $wz \notin E(G)$, as otherwise $\overline{C}$ is obtained from C by replacing the path (w, x, y, z, u) with (w, z, y, x, d, u). Hence, $tw \in E(G)$, as otherwise $\deg_G(w) = 2$. Then $\overline{C}$ is obtained from C by replacing the path (t, v, w, x, y, z, u) with $(t, w, v, a, z, y, x, d, u)$, which contradicts the choice of C.

Case 4h: *f has an opposite minor 3-face f' with middle C-edge vw or yz and an opposite 4-face f_1* (see Figure 15).

Without loss of generality, let yz be the middle C-edge of f'. Then $m_{f,f_1} = 2$, as otherwise wx is a C-edge of a major face, which would imply $w(f) = 4 - 1 + 1 > {}^{10}/_3$. Hence, f_1 sends weight ${}^2/_3$ to f by R4, which implies that f must send weight ${}^2/_3$ to f_1 by R4, as otherwise $w(f) \geq {}^{10}/_3$. Hence, f_1 has an opposite minor 2- or 3-face f_2 that satisfies $m_{f_1,f_2} = 2$ (see Figure 15). Then $wy \notin E(G)$, as otherwise C can be extended by replacing the path (v, w, x, y, z, q) with (v, a, z, y, w, x, d, q), and $wz \notin E(G)$, as otherwise C can be extended by replacing the path (w, x, y, z, q) with (w, z, y, x, d, q).

If f_2 is a 3-face, this implies by symmetry $tw \notin E(G)$ and $uw \notin E(G)$, which contradicts $\deg_G(w) \geq 3$. Hence, f_2 is a 2-face. Then $uw \notin E(G)$, as otherwise C can be extended by replacing the path (t, u, v, w) with (t, g, v, u, w), which implies $tw \in E(G)$, as otherwise $\deg_G(w) = 2$. This contradicts $\deg_G(u) \geq 3$.

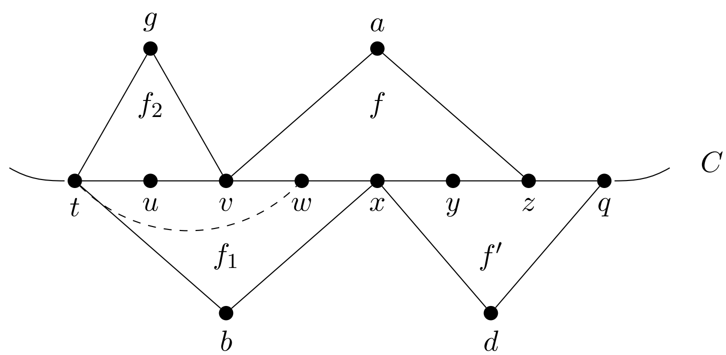

Figure 15: Case 4h

Case 4i: *f has no opposite minor 3-face whose middle C-edge is a C-edge of f* (see Figure 16).

Then, as argued before, f has an opposite minor 4-face f' with $m_{f,f'} = 2$ and C-edges xy and yz, that has an opposite minor 2- or 3-face f_2 with $m_{f',f_2} = 2$. According to R4, f sends weight $2/3$ to f'. Let f'' be the face opposite to f that has C-edge wx. Then f'' must be either a second opposite minor 4-face with $m_{f,f''} = 2$ that has an opposite minor 2- or 3-face f_1 with $m_{f'',f_1} = 2$ (due to R4), or a opposite minor 5-face with $m_{f,f''} = 2$ that has two opposite minor 2-faces (due to R5), as otherwise $w(f) \geq 4 - 2/3 = 10/3$, since f sends no weight to any 2- or 3-face by R2 or R3. Note that $g = a = h$ and $b = d$ are possible.

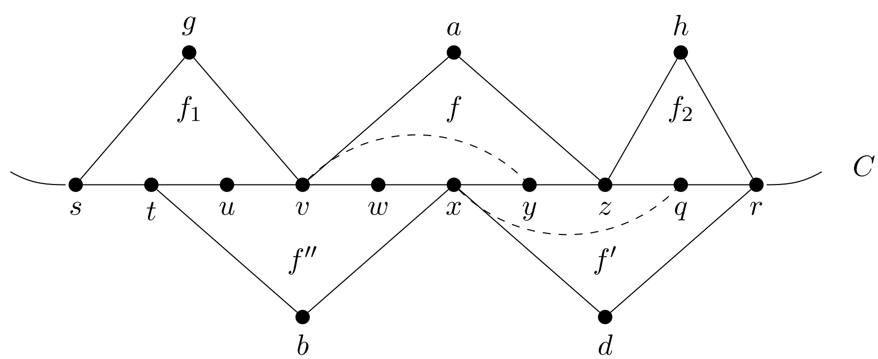

Figure 16: Case 4i

We claim that in all cases vy is an edge of G. Consider the case that f_2 is a 2-face (see Figure 16). Then $yq \notin E(G)$, as otherwise C can be extended by replacing the path (y, z, q, r) with (y, q, z, h, r), and thus $xq \in E(G)$, as otherwise $\deg_G(q) = 2$. This implies that vy or wy is in G, as otherwise $\deg_G(y) = 2$. Since $wy \notin E(G)$, as otherwise C can be extended by replacing the path (w, x, y, z, q, r) with (w, y, x, q, z, h, r), we have $vy \in E(G)$, as claimed. Now consider the remaining case that f_2 is a 3-face. By symmetry, we will assume instead that f_1 is a 3-face and prove that $wz \in E(G)$ (such that the notation of Figure 16 can be used); this implies $vy \in E(G)$ for the case that f_2 is a 3-face. Then $wy \notin E(G)$, as otherwise C can be extended by replacing the path (s,t,u,v,w,x,y) with (s,g,v,u,t,b,x,w,y), and $uw \notin E(G)$, as otherwise C can be extended by replacing the path (s,t,u,v,w,x) with (s,g,v,w,u,t,b,x). In addition, $tw \notin E(G)$, as otherwise C can be extended by replacing the path (s,t,u,v,w) with (s,g,v,u,t,w). Then $wz \in E(G)$, as claimed, since otherwise $\deg_G(w) = 2$, which is a contradiction.

Hence, we proved that in all cases $vy \in E(G)$. If f'' is a 5-face, then $ux \in E(G)$ by the last argument of Case 4e, which contradicts $\deg_G(w) \geq 3$. Hence, f'' is a 4-face, and no matter whether f_1 is a 2- or 3-face, wz is an edge of G by a symmetric argument to the one of the last paragraph. This contradicts that G is plane, because $vy \in E(G)$.

Case 5: *f is a minor 5-face* (see Figure 17).

Then f is initially charged with weight 5. If f looses a total net weight of at most $5/3$, then $w(f) \geq 10/3$, so assume otherwise. We distinguish the following subcases.

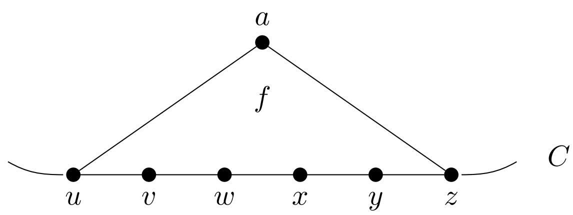

Figure 17: Case 5

Case 5a: *f sends weight to an opposite minor 5-face f'* (see Figure 18).

Without loss of generality, let xy and yz be C-edges of f' by R5. Then f sends weight $1/3$ to f', and f' has two opposite minor 2-faces f_1 and f_2. Since $w(f) < 10/3$, f does neither send weight to a second 5-face nor to a 4-face nor to a 3-face (as there may be at most one of each kind and, if so, no 2-face that receives weight from f). This implies that the edge uv is a C-edge of a minor 2-face f_3 opposite to f, and that vw and wx are the C-edges of a second minor 2-face f_4 opposite to f (see Figure 18). Then f' sends weight $1/3$ back to f by R5, but $w(f) = 5 - 3 \cdot \frac{2}{3} = 3 < 10/3$ is still satisfied.

We have $yp \notin E(G)$ and $pr \notin E(G)$, as otherwise C can be extended by detouring through g. Since $\deg_G(p) \geq 3$, $xp \in E(G)$. By symmetry, $wz \in E(G)$, which implies $yw \in E(G)$. Then C can be extended by replacing the path (v,w,x,y) with (v,b,x,w,y).

Case 5b: *f sends weight to an opposite minor 4-face f'* (see Figure 19).

Without loss of generality, let xy and yz be C-edges of f' by R4. Assume first that f sends weight to an opposite minor 3-face f_1. Then f sends total weight $5/3$ to f' and f_1, and the middle C-edge of f_1 is either uv or vw. Both cases contradict $w(f) < 10/3$, since no further weight is sent. The same argument gives a contradiction if f sends weight to a minor 4-face different from f'.

Hence, f sends a total weight of at least $4/3$ to minor 2-faces, as R2 sends only multiples of weight $2/3$. This implies that f has an opposite minor 2-face f_1 with $m_{f,f_1} = 2$. If f_1 has C-edges uv and vw, then wx is again a C-edge of major face, which sends weight 1 to

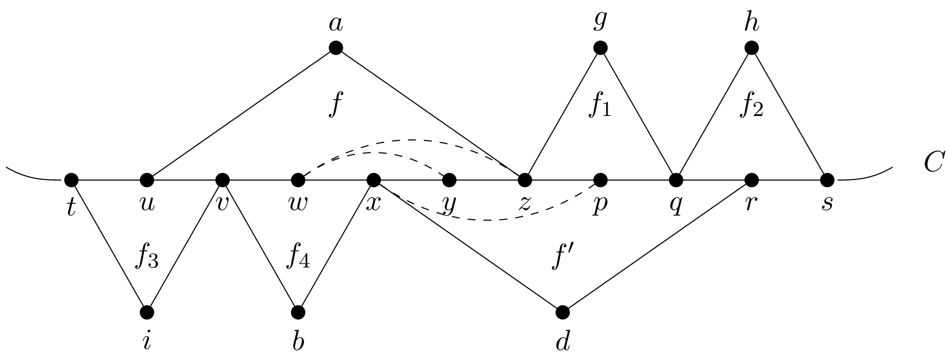

Figure 18: Case 5a

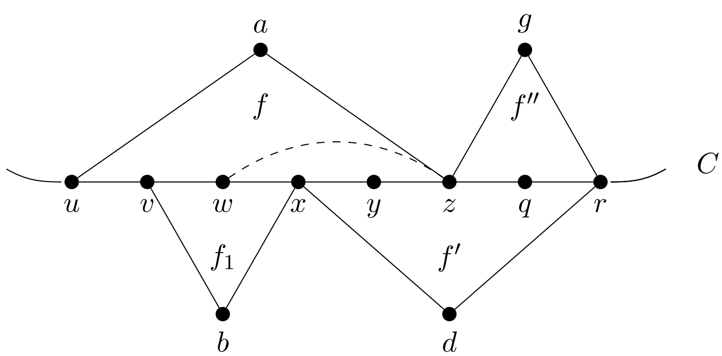

Figure 19: Case 5b

f and thus contradicts $w(f) < {}^{10}/_{3}$. Hence, f_1 has C-edges vw and wx (see Figure 19). Then uw and wy are not edges of G, as otherwise C can be extended by detouring through b. Hence, $wz \in E(G)$, as otherwise $\deg_G(w) = 2$. Moreover, $yq \notin E(G)$ and $xq \in E(G)$ for the same reason as in Case 4i, which contradicts $\deg_G(y) \geq 3$.

Case 5c: *f sends weight to an opposite minor 3-face f' with middle C-edge wx* (see Figure 20).

In order to have $w(f) < {}^{10}/_{3}$, by R1–R3, f sends weight ${}^{2}/_{3}$ to each of the minor 2-faces f_1 and f_2 having C-edges uv and yz, respectively. Then uw and xz are not edges of G, as otherwise C can be extended by detouring C through b or g, respectively. Since $\{v, y\}$ is not a 2-separator of G, this implies that either $wz \in E(G)$ or $ux \in E(G)$, say by symmetry the former. Then we can obtain $\overline{C}$ from C by replacing the path (v, w, x, y, z) with (v, d, y, x, w, z).

Case 5d: *f sends weight to an opposite minor 3-face f' with middle C-edge vw or xy, but not to any opposite minor 4- or 5-face* (see Figure 21).

Without loss of generality, let the middle C-edge of f' be xy. Then $vy \notin E(G)$, as otherwise C can be extended by replacing the path (v, w, x, y, z) with (v, y, x, w, d, z). Let f_1 be the face opposite to f that has vw as a C-edge. Since $w(f) < {}^{10}/_{3}$, f_1 is either a minor 3-face with middle C-edge uv or a minor 2-face with C-edges vw and wx. Assume to the contrary that f_1 is a 2-face. Then $vx \notin E(G)$, as otherwise C can be extended by detouring through b. This implies $vz \in E(G)$, as otherwise $\deg_G(v) = 2$. Then $\{w, z\}$ is

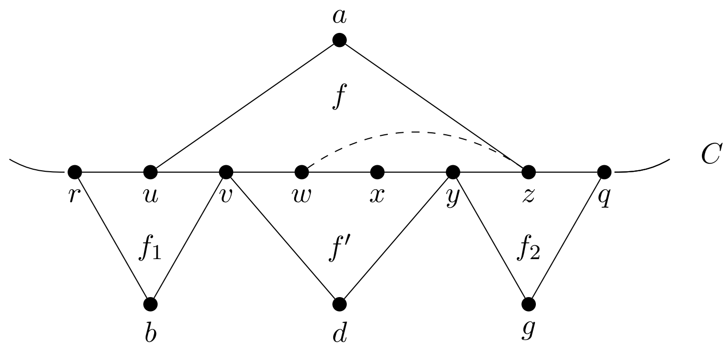

Figure 20: Case 5c

a 2-separator of G, which is a contradiction.

Hence, f_1 is a 3-face (see Figure 21). Then $ux \notin E(G)$, as otherwise C can be extended by replacing the path (r,u,v,w,x) with (r,b,w,v,u,x). Thus, since $\{w,z\}$ is no 2-separator of G, uy or vx is an edge of G. Assume to the contrary that $uy \notin E(G)$. Then $vx \in E(G)$, and we have $wy \notin E(G)$, as otherwise C can be extended by replacing the path (r,u,v,w,x,y,z) with (r,b,w,y,x,v,u,a,z). Since $\deg_G(y) \geq 3$, this implies $uy \in E(G)$. Assume to the contrary that $vx \notin E(G)$. Then $xz \in E(G)$, as otherwise $\deg_G(x) = 2$, and C can be extended by replacing the path (r,u,v,w,x,y,z) with (r,b,w,v,u,y,x,z), which gives a contradiction. Hence, $uy \in E(G)$ and $vx \in E(G)$. Then C can be extended by replacing the path (u,v,w,x,y,z) with (u,y,x,v,w,d,z).

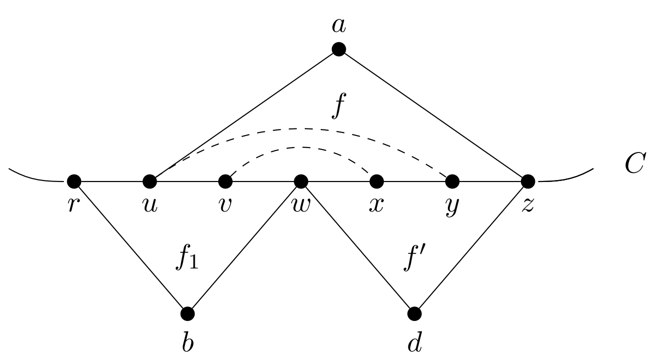

Figure 21: Case 5d

Case 5e: *f sends weight to an opposite minor 3-face f' with middle C-edge uv or yz, but not to any opposite minor 4- or 5-face* (see Figure 22).

Without loss of generality, let the middle C-edge of f' be yz. Assume first that f sends weight to a second opposite minor 3-face $f_1 \neq f'$. By Case 5d, f_1 has not middle C-edge vw, so that f' must have middle C-edge uv. Then wx is a C-edge of a major face opposite to f that sends weight 1 to f, which contradicts $w(f) < 10/3$.

Hence, in order to satisfy $w(f) < 10/3$, f sends by R2 a total weight of $4/3$ to opposite minor 2-faces. This implies that there is a minor 2-face f_2 opposite to f that satisfies $m_{f,f_2} = 2$. Then f_2 has not C-edges uv and vw, as otherwise wx would once again be a C-edge of a major face, which contradicts $w(f) < 10/3$. Hence, f_2 has C-edges vw and

wx (see Figure 22). Then $uw \notin E(G)$, as otherwise C can be extended by replacing the path (u, v, w, x) with (u, w, v, b, x), and $wy \notin E(G)$, as otherwise C can be extended by replacing the path (v, w, x, y) with (v, b, x, w, y). Since $\deg_G(w) \geq 3$, $wz \in E(G)$. Then C can be extended by replacing the path (w, x, y, z, q) with (w, z, y, x, d, q), which is a contradiction.

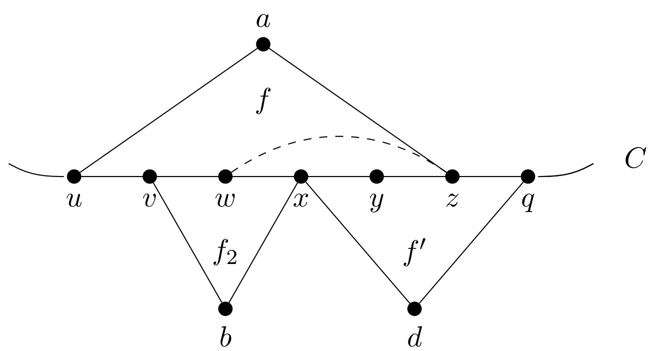

Figure 22: Case 5e

We conclude that f sends no weight to any opposite minor 3-, 4- or 5-face. In order to satisfy $w(f) < 10/3$, f must therefore send a total weight of $6/3$ to opposite minor 2-faces by R2. In particular, there is at least one minor 2-face f' opposite to f that has $m_{f,f'} = 2$. We distinguish the following subcases for f'.

Case 5f: *f' has C-edges uv and vw, or xy and yz* (see Figure 23).

Without loss of generality, let f' have C-edges xy and yz. Assume first that f has a second opposite minor 2-face $f_1 \neq f'$ with $m_{f,f_1} = 2$. Then f_1 has not C-edges uv and vw, as then wx would be a C-edge of a major face sending f weight 1, which implies $w(f) = 5 - 4 \cdot \frac{2}{3} + 1 = 10/3$. Hence, f_1 has C-edges vw and wx (see Figure 23). Then $wy \notin E(G)$, as otherwise C can be extended by replacing the path (w, x, y, z) with (w, y, x, d, z). Hence, $vy \notin E(G)$, as otherwise $\deg_G(w) = 2$. Since $\deg_G(y) \geq 3$, we conclude $uy \in E(G)$ and, by $\deg_G(w) \geq 3$, $uw \in E(G)$. Then C can be extended by replacing the path (u, v, w, x) with (u, w, v, b, x).

Hence, f has no second opposite minor 2-face $f_1 \neq f'$ with $m_{f,f_1} = 2$. Since f sends a total weight of $6/3$ to opposite minor 2-faces by R2, f has an opposite minor 2-face $f_2 \neq f'$ that has C-edge uv but no other C-edge of f. Then vw and wx are C-edges of major face(s), which contradicts $w(f) < 10/3$.

Case 5g: *f' has C-edges vw and wx, or wx and xy* (see Figure 24).

Without loss of generality, let f' have C-edges wx and xy. By Case 5f, f has no second opposite minor 2-face $f_1 \neq f'$ with $m_{f,f_1} = 2$. By $w(f) < 10/3$, f has an opposite minor 2-face f_2 that has exactly one of the C-edges of f as a C-edge. If this edge e is not yz, $e = uv$ and then vw is a C-edge of a major face, which contradicts $w(f) < 10/3$. Hence $e = yz$. Since neither uv nor vw is a C-edge of a major face, as this would again contradict $w(f) < 10/3$, uv and vw are C-edges of a minor j-face f_3 with $j \geq 4$ that does not receive any weight from f. Then f_3 sends weight $1/3$ to f by R5, which gives $w(f) = 10/3$ and thus a contradiction.

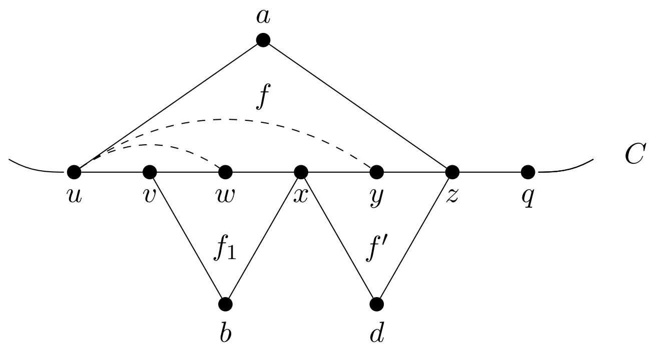

Figure 23: Case 5f

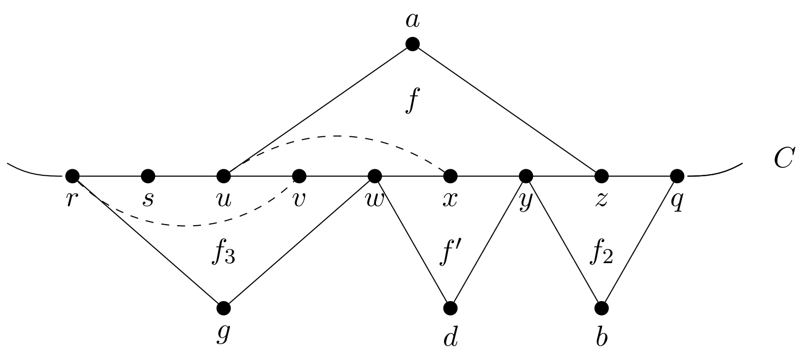

Figure 24: Case 5g

Case 6: *f is a minor* 6*-face* (see Figure 25).

Then f is initially charged with weight 6. If f looses a total net weight of at most $8/3$, then $w(f) \geq 10/3$, so assume that f looses a total net weight of at least $9/3$. We distinguish the following subcases.

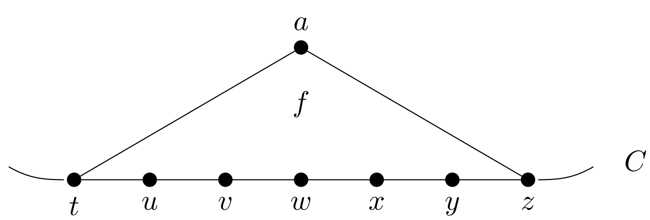

Figure 25: Case 6

Case 6a: *f sends weight to an opposite minor 5-face f'* (see Figure 26).

Without loss of generality, let xy and yz be C-edges of f' getting weight from f by R5. Then f sends weight $1/3$ to f', and total weight $8/3$ to opposite minor 2-faces f_3 and f_4 by R1–R5, as otherwise $w(f) \geq 10/3$ (see Figure 26). Let f_1 and f_2 be the two minor 2-faces opposite to f' due to R5.

We have $uw \notin E(G)$ and $wy \notin E(G)$, as otherwise C can be extended by detouring

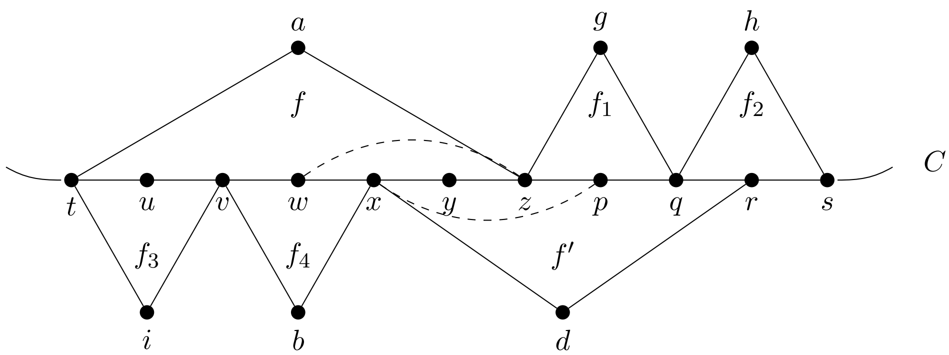

Figure 26: Case 6a

through b, and $tw \notin E(G)$, as otherwise $\deg_G(u) = 2$. Since $\deg_G(w) \geq 3$, $wz \in E(G)$. Moreover, $yp \notin E(G)$ and $pr \notin E(G)$, as otherwise C can be extended by detouring through g. Since $\deg_G(p) \geq 3$, $xp \in E(G)$. Hence, $\deg_G(y) = 2$, which contradicts that G is 3-connected.

Case 6b: *f sends weight to an opposite minor 4-face f'* (see Figure 27)*.*

Without loss of generality, let xy and yz be C-edges of f' by R4. Since $w(f) < 10/3$, f has neither an opposite minor 5-face, nor a second opposite minor 4-face. Assume first that f sends weight to an opposite minor 3-face f_1. Then f sends total weight $5/3$ to f' and f_1, and must therefore send weight $4/3$ to minor 2-face(s), as otherwise $w(f) \geq 10/3$. Hence, f_1 has middle C-edge tu, and f has one opposite minor 2-face f_2 that has C-edges vw and wx (see Figure 27).

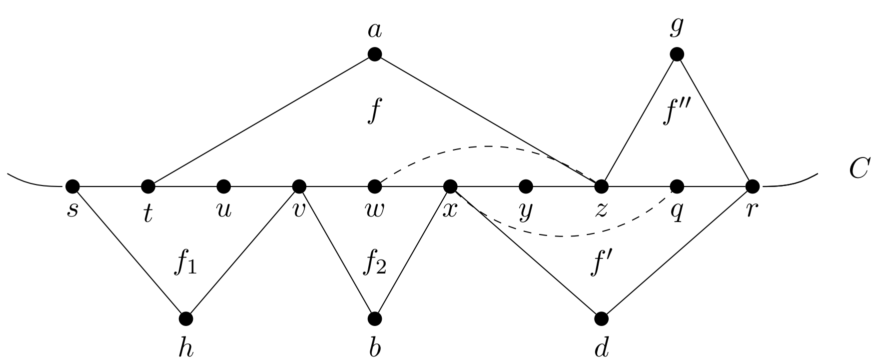

Figure 27: Case 6b

Then uw and wy are not edges of G, as otherwise C can be extended by detouring through b. Moreover, $tw \notin E(G)$, as otherwise C can be extended by replacing the path (s, t, u, v, w) with (s, h, v, u, t, w). Hence, $wz \in E(G)$, as otherwise $\deg_G(w) = 2$. Moreover, $yq \notin E(G)$ and $xq \in E(G)$ for the same reason as in Case 4i, which contradicts $\deg_G(y) \geq 3$.

Case 6c: *f sends weight to an opposite minor 3-face f' with middle C-edge vw or wx* (see Figure 28)*.*

Without loss of generality, let the middle C-edge of f' be wx. In order to have $w(f) < 10/3$, f must by R2–R3 send weight 2 to minor 2-faces. Thus, f has two minor 2-faces

f_1 and f_2 such that f_1 has C-edges tu and uv, and f_2 has yz as a C-edge.

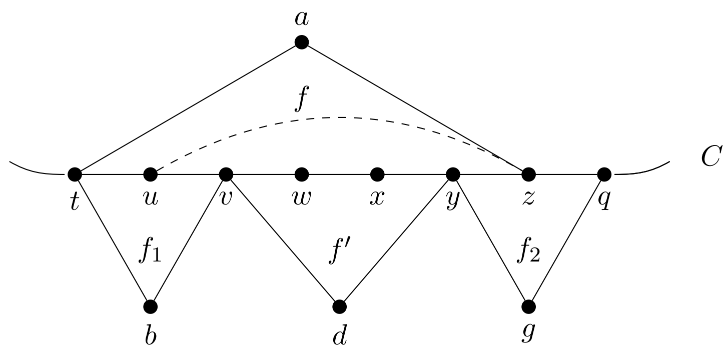

Figure 28: Case 6c

Then $uw \notin E(G)$, as otherwise C can be extended by detouring C through b. In addition, $ux \notin E(G)$, as otherwise C can be extended by replacing the path (u, v, w, x, y) with (u, x, w, v, d, y). Then $uy \notin E(G)$, as otherwise the fact that $\{v, y\}$ is not a 2-separator of G would imply $uw \in E(G)$ or $ux \in E(G)$. Since $\deg_G(u) \geq 3$, $uz \in E(G)$. Then we can obtain $\overline{C}$ from C by replacing the path (t, u, v, w, x, y, z, q) with $(t, a, z, u, v, w, x, y, g, q)$.

Case 6d: *f sends weight to an opposite minor 3-face f' with middle C-edge uv or xy* (see Figure 29).

Without loss of generality, let the middle C-edge of f' be xy. As in Case 6c, $w(f) < 10/3$ implies that f has opposite minor 2-faces f_1 and f_2 such that f_2 has C-edges uv and vw and f_1 has C-edge tu (see Figure 29).

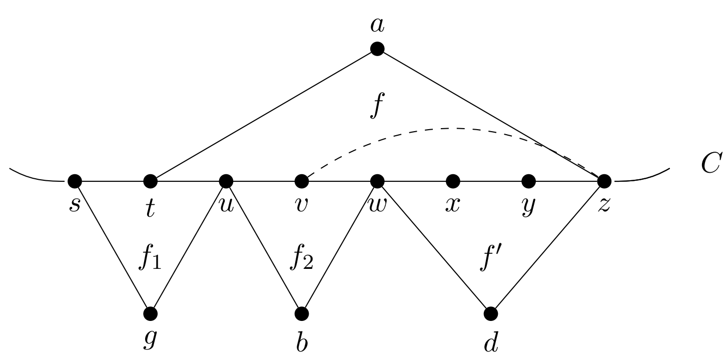

Figure 29: Case 6d

Then tv and vx are not edges of G, as otherwise C can be extended by detouring C through b. In addition, $vy \notin E(G)$, as otherwise C can be extended by replacing the path (v, w, x, y, z) with (v, y, x, w, d, z). Since $\deg_G(v) \geq 3$, $vz \in E(G)$. This implies that $\{w, z\}$ is a 2-separator of G, which contradicts that G is 3-connected.

Case 6e: *f sends weight to an opposite minor 3-face f' with middle C-edge tu or yz, but not to any opposite minor 4- or 5-face* (see Figure 30).

Without loss of generality, let the middle C-edge of f' be yz. Assume first that f has a second opposite minor 3-face f''. By Cases 6c+d, f'' has middle C-edge tu. By $w(f) < 10/3$, f has an opposite minor 2-face f_2 with C-edges vw and wx (see Figure 30). Then

$uw \notin E(G)$ and $wy \notin E(G)$, as otherwise C can be extended by detouring through b. Moreover, $wz \notin E(G)$, as otherwise C can be extended by replacing the path (w, x, y, z, q) with (w, z, y, x, d, q). By symmetry, $tw \notin E(G)$, which contradicts $\deg_G(w) \geq 3$.

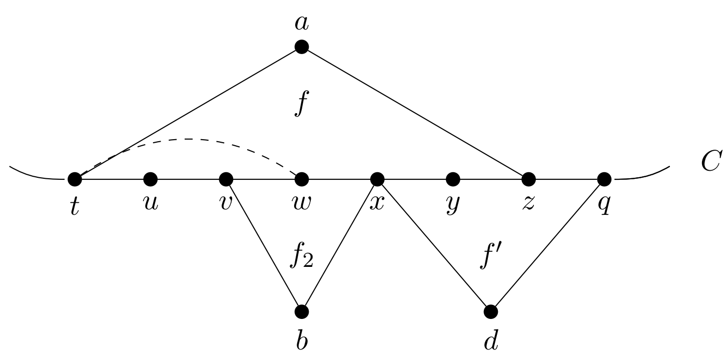

Figure 30: Case 6e

Hence, by R1–R3, f sends total weight 2 to at least two opposite minor 2-faces f_1 and f_2. If $m_{f,f_1} = 1$ or $m_{f,f_2} = 1$, either the edge uv or the edge wx would be a C-edge of a major face, which contradicts $w(f) < 10/3$. Thus, f_1 has C-edges tu and uv, and f_2 has C-edges vw and wx. From the previous argument, we know that uw, wy and wz are not in G. Since $\deg_G(w) \geq 3$, $tw \in E(G)$. This contradicts $\deg_G(u) \geq 3$.

We conclude that f sends no weight to any opposite minor 3-, 4- or 5-face. In order to satisfy $w(f) < 10/3$, f must therefore send a total weight of $10/3$ to opposite minor 2-faces by R2, as R2 sends only multiples of weight $2/3$. If some C-edge e of f is not a C-edge of a minor 2-face, e must be either tu or yz, as otherwise e would be in a major face that sends weight 1 to f and therefore contradicts $w(f) < 10/3$. Hence, f has three opposite minor 2-faces f_1, f_2 and f_3 such that $m_{f,f_1} = m_{f,f_2} = 2$ and the C-edges of f_1 and f_2 are either uv, vw, wx, xy or one of tu, uv, vw, wx and vw, wx, xy, yz. We distinguish these subcases.

Case 6f: *The C-edges of f_1 and f_2 are tu, uv, vw, wx or vw, wx, xy, yz* (see Figure 31).

Without loss of generality, let f_1 and f_2 have the C-edges vw, wx, xy, yz. By the above argument, f_3 has the C-edges tu and uv (see Figure 31).

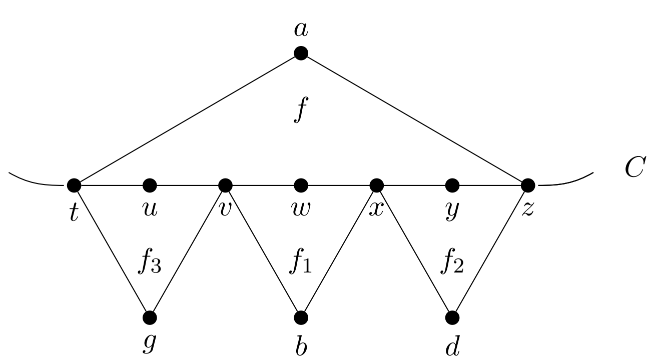

Figure 31: Case 6f

Then uw and wy are not in G, as otherwise C can be extended by detouring through b. Moreoever, $wz \notin E(G)$, as otherwise $\deg_G(y) = 2$. By symmetry, $tw \notin E(G)$, which contradicts $\deg_G(w) \geq 3$.

Case 6g: *The C-edges of f_1 and f_2 are uv, vw, wx, xy* (see Figure 32).

Then f_3 has either tu or yz as a C-edge, say without loss of generality the latter.

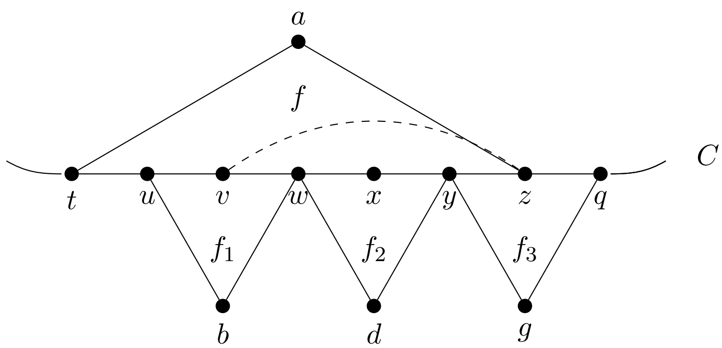

Figure 32: Case 6g

Then tv and vx are not in G, as otherwise C can be extended by detouring through b. Moreover, $vy \notin E(G)$, as otherwise $\deg_G(x) = 2$. Since $\deg_G(v) \geq 3$, $vz \in E(G)$. Then $xz \notin E(G)$, as otherwise C can be extended by detouring through g. Hence, we obtain the contradiction $\deg_G(x) = 2$.

Case 7: *f is a minor 7-face* (see Figure 33).

Then f is initially charged with weight 7. If f looses a total net weight of at most $11/3$, then $w(f) \geq 10/3$, so assume that f looses a total net weight of at least $12/3$. According to R1–R5, f sends to every opposite face f' at most weight $\frac{2}{3}m_{f,f'}$ (for example, if f' is a minor 3-face, f sends only weight at most $\frac{1}{2}m_{f,f'}$ by R3). Hence, f does not send any weight to a 5-face, as otherwise $w(f) \geq 10/3$. We distinguish the remaining cases.

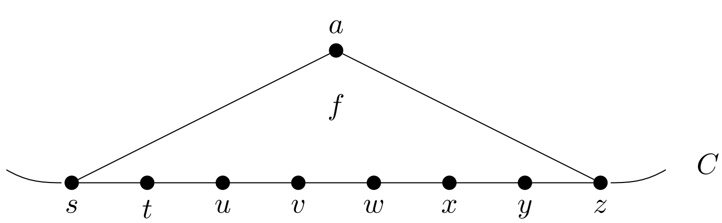

Figure 33: Case 7

Case 7a: *f sends weight to an opposite minor 4-face f'* (see Figure 34).

Without loss of generality, let f' have C-edges xy and yz. Since $w(f) < 10/3$, all other C-edges of f are C-edges of minor 2-faces f_1, f_2 and f_3 (see Figure 34).

Then $yp \notin E(G)$, as otherwise C can be extended by detouring through g, and hence $xp \in E(G)$, as otherwise $\deg_G(p) = 2$. Also, uw and wy are not in G, as otherwise C can be extended by detouring through b. Hence, y has a neighbor in G that is incident to f and

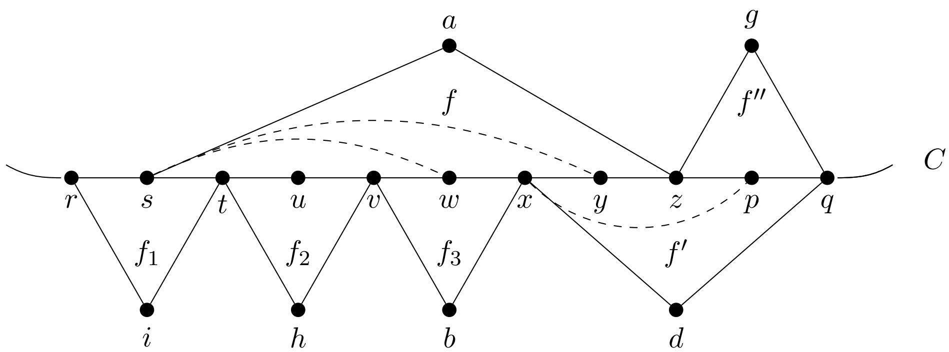

Figure 34: Case 7a

different from $\{w, x, z\}$. We conclude $wz \notin E(G)$. In addition, $tw \notin E(G)$, as otherwise $\deg_G(u) = 2$. Thus, $sw \in E(G)$, which implies $sy \in E(G)$. Then $\overline{C}$ can be obtained from C by replacing the path $(r, s, t, u, v, w, x, y, z)$ with $(r, i, t, u, v, w, x, y, s, a, z)$.

Case 7b: *f sends weight to an opposite minor 3-face f'* (see Figure 35).

Since $w(f) < 10/3$, the middle C-edge of f' must be either st or yz; say without loss of generality the latter. For the same reason as in Case 7a, all other C-edges of f are C-edges of minor 2-faces f_1, f_2 and f_3 (see Figure 35). Note that if there is another 3-face f'' with middle C-edge st, then the edges uv, vw and wx are not all C-edges of some 2-face.

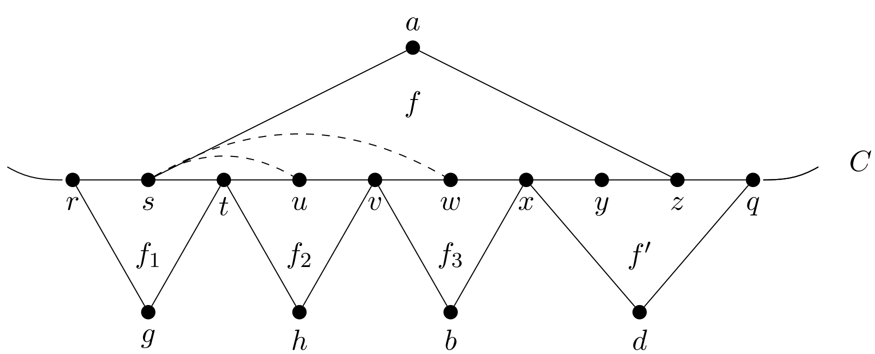

Figure 35: Case 7b

Then $uw \notin E(G)$ and $wy \notin E(G)$, as otherwise C can be extended by detouring through b. Moreover, $wz \notin E(G)$, as otherwise C can be extended by replacing the path (w, x, y, z, q) with (w, z, y, x, d, q). Also $tw \notin E(G)$, as otherwise $\deg_G(u) = 2$. Since $\deg_G(w) \geq 3$, $sw \in E(G)$. Since $\deg_G(u) \geq 3$, $su \in E(G)$. Then C can be extended by replacing the path (s, t, u, v) with (s, u, t, h, v).

Case 7c: *f sends no weight to 3-, 4- and 5-faces* (see Figure 36).

Then f sends a total weight of at least $6 \cdot \frac{2}{3} = 4$ to opposite minor 2-faces. The C-edges of these 2-faces must be consecutive on C, as otherwise exactly one C-edge of f would be a C-edge of a major face, which contradicts $w(f) < 10/3$. Hence, there are three minor 2-faces f_1, f_2 and f_3, whose C-edges are consecutive on C and satisfy

$m_{f,f_1} = m_{f,f_2} = m_{f,f_3} = 2$ (see Figure 36). Assume without loss of generality that f_3 has C-edges xy and yz.

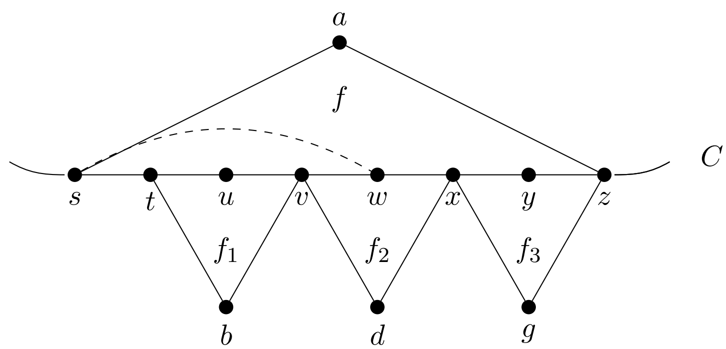

Figure 36: Case 7c

Then uw and wy are not in G, as otherwise C can be extended by detouring through d. Moreover, tw and wz are not in G, as otherwise $\deg_G(u) = 2$ or $\deg_G(y) = 2$. Since $\deg_G(w) \geq 3$, $sw \in E(G)$. Moreover, $su \notin E(G)$, as otherwise C can be extended by detouring through b. Hence, we obtain the contradiction $\deg_G(u) = 2$.

Case 8: *f is a minor 8-face* (see Figure 37).

Then f is initially charged with weight 8. If f looses a total net weight of at most $^{14}/_{3}$, then $w(f) \geq {}^{10}/_{3}$, so assume that f looses a total net weight of at least $^{15}/_{3}$. Hence, f does not send any weight to a 4- or 5-face, as otherwise $w(f) \geq {}^{10}/_{3}$. We distinguish the remaining cases.

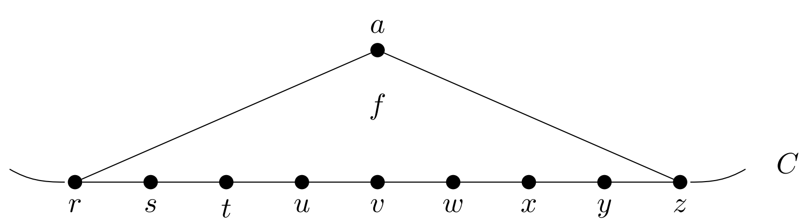

Figure 37: Case 8

Case 8a: *f sends weight to an opposite minor 3-face f'* (see Figure 38).

Then $w(f) < {}^{10}/_{3}$ implies that f' has exactly two C-edges that are C-edges of f, and that every other C-edge of f is a C-edge of a minor 2-face. Without loss of generality, let f' have middle C-edge yz, and let f_1, f_2 and f_3 be the minor 2-faces opposite to f (see Figure 38).

Then su, uw and wy are not edges of G, as otherwise C can be extended by detouring through h or b. Moreover, $wz \notin E(G)$, as otherwise C can be extended by replacing the path (w, x, y, z, q) with (w, z, y, x, d, q). Also $sw \notin E(G)$ and $tw \notin E(G)$, as otherwise $\deg_G(u) = 2$. Since $\deg_G(w) \geq 3$, $rw \in E(G)$. Since $\deg_G(u) \geq 3$, $ru \in E(G)$. This gives the contradiction $\deg_G(s) = 2$.

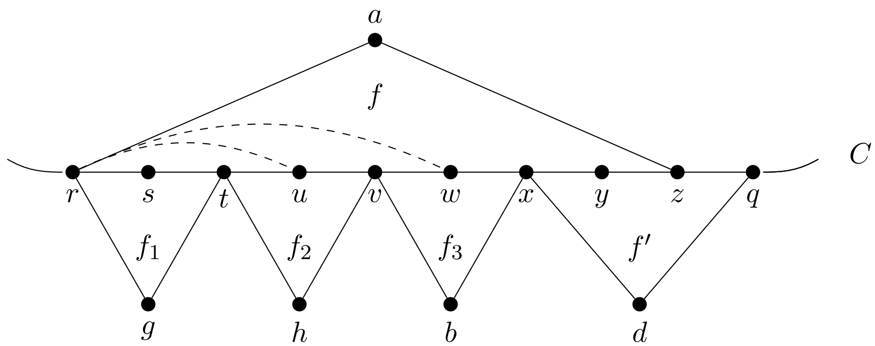

Figure 38: Case 8a

Case 8b: *f sends no weight to 3-, 4- and 5-faces* (see Figure 39).

Then f sends a total weight of exactly $8 \cdot \frac{2}{3} = {}^{16}/_{3}$ to opposite minor 2-faces, as R2 sends only multiples of $\frac{2}{3}$ weight. Assume first that a minor 2-face f_4 opposite to f has C-edges xy and yz (see Figure 39). Then $wy \notin E(G)$, as otherwise C can be extended by detouring through g, and $wz \notin E(G)$, as otherwise $\deg_G(y) = 2$. Then the same arguments as in Case 8a give the contradiction $\deg_G(s) = 2$.

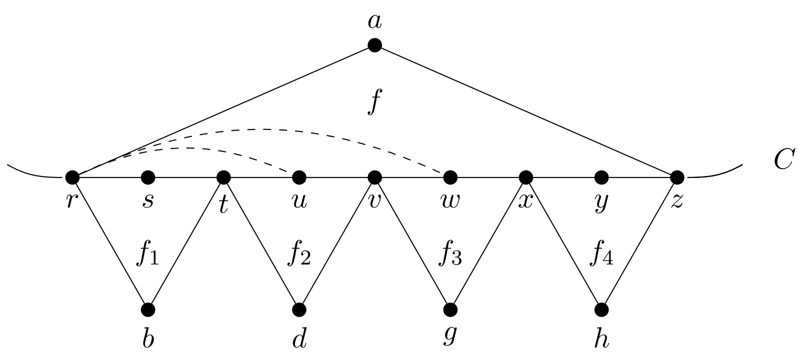

Figure 39: Case 8b

Hence, let yz be the only C-edge of f_4 that is a C-edge of f. Then v has no neighbor that is incident to f and not in $\{u, w\}$, as otherwise t or x has degree 2 in G. Hence, we obtain the contradiction $\deg_G(v) = 2$.

Case 9: *f is a minor j-face with $j \geq 9$* (see Figure 40).

Then f is initially charged with weight j and looses a total net weight of at most $\frac{2}{3}j$, so that $w(f) \geq \frac{1}{3}j \geq \frac{10}{3}$ if $j \geq 10$. Hence, $j = 9$ and every C-edge of f is a C-edge of a minor 2-face. Since 9 is odd, we may assume without loss of generality that one minor 2-face f_1 has qr but no other C-edge of f as a C-edge (see Figure 40). Then the same arguments as in Cases 8a+b imply that $\deg_G(s) = 2$.

This proves $2c = \sum_{f \in F(H)} w(f) \geq {}^{10}/_{3} \cdot |M^- \cup M^+|$, which completes the proof of Theorem 1. $\square$

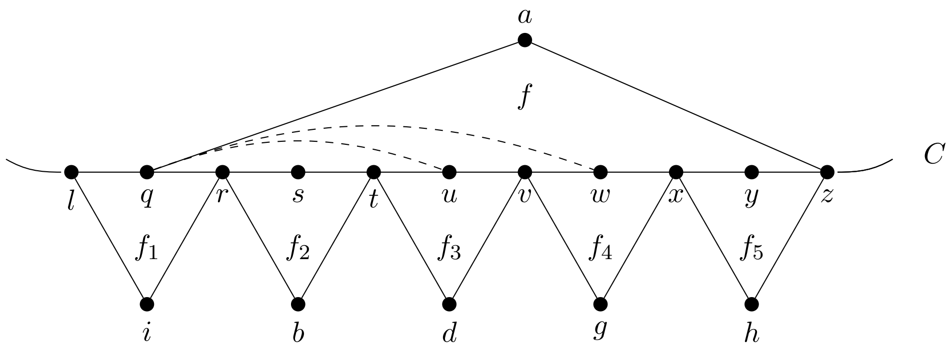

Figure 40: Case 9

4 Remarks

We remark that the bound of Theorem 1 can be improved to $\frac{5}{8}(n+4)$ for every $n \geq 16$: then Lemma 5 in [2] implies the improved bound for the special case that V^- or V^+ is empty, while in the remaining case $|V^-| \geq 1 \leq |V^+|$ Lemma 2 can be immediately strengthened to $|M^- \cup M^+| \geq |V^- \cup V^+| + 4$ using the same proof with a different induction base (see also [3]). This immediately improves the bound $\mathrm{circ}(G) \geq \frac{13}{21}(n+4)$ given in [2] for every $n \geq 16$. We note that $\mathrm{circ}(G) \geq \frac{5}{8}(n+4)$ does not hold for $n \leq 6$, as for these values a cycle of length at least $\frac{5}{8}(n+4) > n$ is impossible.

The proof of Theorem 1 is constructive and gives a quadratic-time algorithm that finds a cycle of length at least $\frac{5}{8}(n+2)$, by applying the result of [6] exactly as shown in [3, Section Algorithm]. We therefore conclude the following theorem.

Theorem 3. *For every essentially 4-connected plane graph G on n vertices, a cycle of length at least $\frac{5}{8}(n+2)$ can be computed in time $O(n^2)$.*

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