Lower bounds for locally highly connected graphs *

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Abstract

We propose a conjecture regarding the lower bound for the number of edges in locally k-connected graphs and we prove it for $k = 2$. In particular, we show that every connected locally 2-connected graph is M_3-rigid. For the special case of surface triangulations, this fact was known before using topological methods. We generalize this result to all locally 2-connected graphs and give a purely combinatorial proof.

Our motivation to study locally k-connected graphs comes from lower bound conjectures for flag triangulations of manifolds, and we discuss some more specific problems in this direction.

1 Introduction

In this note we study graph connectivity from a local perspective. A simple graph $G = (V, E)$ is k-connected, for $k \geq 1$, if $G - W$ is non-empty and connected for every set $W \subseteq V$ with $|W| \leq k - 1$.[1] For a vertex $v \in V$ the *link* of v in G, denoted $\mathrm{lk}_G v$, is the subgraph of G induced by the

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[1] According to our definition the complete graph K_k is k-connected, which is not the case if one uses the alternative definition via k vertex-disjoint paths between every pair of vertices.

open neighborhood of v (our graphs are simple and loop-less, so the open neighborhood of v does not contain v).

If $\mathcal{P}$ is any graph property then we say a graph G *has the property* $\mathcal{P}$ *locally*, in short G *is locally* $\mathcal{P}$, if for every vertex $v \in V$ the link $\mathrm{lk}_G v$ of v satisfies $\mathcal{P}$. Various well-studied graph properties have local descriptions; for instance triangle-free graphs are locally edge-less and quasi-line graphs are defined as locally co-bipartite. Local graph properties of the kind concerned in this paper are notoriously difficult to study because of the possibly intricate ways the vertex neighborhoods can intersect. The purpose of this note is to propose the following conjecture and study some of its special cases.

Conjecture. For every $k \geq 1$ there exists a constant $F(k)$ for which the following holds. For any connected, locally k-connected graph G,

$$|E(G)| \geq (k+1)|V(G)| - F(k) \ .$$

It is known that $F(1) = 3$, that is every n-vertex, connected, locally connected graph has at least $2n-3$ edges [12]. Our main result, Theorem 5, gives the lower bound of $3n - 6$ for the number of edges in a connected, locally 2-connected graph on n vertices. That establishes $F(2) = 6$. For $k \geq 3$ the conjecture, as of now, is open.

Note that if G is connected, locally k-connected and different from K_{k+1}, then every vertex has degree at least $k+1$, so trivially G satisfies $|E| \geq \frac{k+1}{2}|V|$. Our conjecture is then an attempt to improve this bound by roughly a factor of 2. The conjecture is the strongest possible, in the sense that the linear term $(k+1)|V|$ cannot be replaced with $(k+1+\varepsilon)|V|$ for any $\varepsilon > 0$, see Example 6.

If G is a graph of a triangulation of a surface (connected, without boundary) then the link of every vertex in G is Hamiltonian, so in particular G is connected and locally 2-connected. In this case the lower bound $|E(G)| \geq 3|V(G)| - 6$ is a classical consequence of the Euler characteristic formula. Local 2-connectedness can be seen as a good graph-theoretic generalization, which leads to a purely combinatorial proof of the same bound. In fact we will take this analogy even further, showing that a connected, locally 2-connected graph is M_3-rigid (Definition 1) in the sense of Tay [8], generalizing the same statement for graphs of surfaces [8].

In Sect. 4 we propose further conjectures and briefly explain the motivation behind questions of this kind, which are inspired by the study of flag manifold triangulations.

Related work on local connectivity. Zelinka [12] showed $F(1) = 3$. Structural properties of locally 1-connected and (globally) m-connected graphs for various m were studied at depth by Kriesell [6]. Some sufficient conditions for a graph to be locally k-connected are given by Chartrand and Pippert [3]. Not much is known about lower bounds, except for the result of Vanderjagt [9], who finds lower bounds for the number of vertices in a m-connected, locally k-connected graph.

Borowiecki, Borowiecki, Sidorowicz and Skupień [2] show that $|E(G)| \geq (k+1)|V(G)| - \binom{k+2}{2}$ holds for graphs G that are locally k-trees (A *k-tree* is a graph formed from the complete graph K_k by repeatedly adding vertices so that the neighborhood of every new vertex is a k-clique.). Every k-tree is k-connected, so Conjecture 1 holds for the smaller class of locally k-trees. We will discuss the relation between the resulting constant terms in the last section.

2 Preliminaries

Let us first establish additional notation. For a graph G, let $V(G)$ denote its set of vertices and $E(G)$ its set of edges. Let G be a graph. For subsets $W \subseteq V(G)$, let $G[W]$ denote the subgraph of G induced by W. To simplify notation we write $G - W$ for $G[V(G) \setminus W]$ and $G - v$ for $G - \{v\}$. For $v \in V(G)$ let $N_G(v) = \{w \in V \ : \ vw \in E\}$ and $N_G[v] = N_G(v) \cup \{v\}$. We define the *degree* and *link* of v in G as $\deg_G(v) = |N_G(v)|$ and $\mathrm{lk}_G v = G[N_G(v)]$. A graph H is a *spanning* subgraph of G if $V(H) = V(G)$ and $E(H) \subseteq E(G)$.

We use standard symbols for special graphs: P_n for a path, C_n for a cycle and K_n for a complete graph with n vertices. If H is a graph on at most n vertices then $K_n - H$ denotes the graph obtained from K_n by removing a set of edges which forms a subgraph isomorphic to H.

For $k \geq 1$, we define

$$\beta_k(G) = |E(G)| - (k+1)|V(G)| \ . \tag{1}$$

For vertex sets $W \subseteq V(G)$ we denote by $E(G, W)$ the subset of $E(G)$ consisting of those edges which have at least one end in W. The following definition is adapted from Tay [8] and Whiteley [11].

Definition 1. For $d \in \mathbb{N}$, a graph G is *M_d-rigid* if for every set $W \subseteq V(G)$ for which $|V(G) \setminus W| \geq d$, we have $|E(G, W)| \geq d|W|$.

It is easy to see that an M_d-rigid graph G which contains at least one clique of size d satisfies the *lower bound*

$$|E(G)| \geq d|V(G)| - \binom{d+1}{2}, \tag{2}$$

i.e., $\beta_{d-1}(G) \geq -\binom{d+1}{2}$. Indeed, if $C = \{v_1, \ldots, v_d\}$ is a clique then

$$|E(G)| = |E(G, V(G) \setminus C)| + \binom{d}{2} \geq d(|V(G)| - d) + \binom{d}{2} = d|V(G)| - \binom{d+1}{2} .$$

3 Locally 2-connected graphs

In this section we establish Conjecture 1 for $k = 2$. To show that a locally 2-connected graph G cannot be too sparse we employ a recursive approach, which we now outline. If the degree of every vertex in G is at least 6 then we have even more than we need, since then $|E(G)| \geq 3|V(G)|$. Otherwise, we choose a vertex v of degree at most 5 and remove it. This can affect the 2-connectivity of certain links, but we fix it by adding some new edges between the vertices of $N_G(v)$. If the number of new edges is not too large then the whole process does not increase β_2 and induction follows. This reduction step takes place in Proposition 4, after which the main result of this section, Theorem 5, is a simple consequence.

We proceed with the technical lemmas used to describe the modifications of the graph. The next lemma is classical (a proof of its second claim was also given by West [10, Lemma 4.2.3]).

Lemma 2. *Let $k \geq 1$, let G be a graph and let $v \in V(G)$.*

a) If G and $\mathrm{lk}_G v$ are k-connected, then $G - v$ is k-connected.

b) If $\deg_G(v) \geq k$ and $G - v$ is k-connected, then G is k-connected.

Proof. a) The claim is obvious when $k = 1$. Now suppose that $k \geq 2$ and let $W \subseteq V(G) \setminus \{v\}$ be any set with $|W| \leq k - 1$. The graphs $G - W$ and $\mathrm{lk}_{G-W} v = \mathrm{lk}_G v - W$ are connected by assumption. By the claim for 1-connectivity, the graph $(G - W) - v = (G - v) - W$ is connected. Since the choice of W was arbitrary, we get that $G - v$ is k-connected.

b) Let $W \subseteq V(G)$ be any set with $|W| \leq k - 1$. If $v \in W$ then $G - W = (G - v) - (W \setminus \{v\})$ is non-empty and connected, by the k-connectivity of $G - v$. If $v \notin W$ then $(G - v) - W$ is non-empty and connected, and moreover $\deg_{G-W}(v) \geq 1$. This implies that $G - W$ is connected. Since the choice of W was arbitrary, we obtain that G is k-connected. $\square$ $\square$

We will say that a graph transformation is 2-*admissible* if it transforms a 2-connected graph into a 2-connected graph. For clarity, we remind the reader that according to our definition the graph K_2 is 2-connected.

Lemma 3. *The following graph operations are 2-admissible:*

- `del`(v)*: remove a vertex* v *whose link is* 2*-connected.*
- `add`$(v; v_1, \dots, v_s)$*: add a new vertex* v *adjacent to existing vertices* $v_1, \dots, v_s$ *with* $s \geq 2$*.*
- `add`(xy)*: add a new edge between existing vertices* x*,* y*.*
- $v \to x$*: rename an existing vertex* v *with a new label* x*, not used for any other vertex of the graph.*
- `sub`$(xy; v)$*: subdivide the edge* xy *with a new vertex* v*, as long as the initial graph had at least* 3 *vertices.*

Proof. The transformations `del`(v) and `add`$(v; v_1, \dots, v_s)$ are 2-admissible by Lemma 2. For `add`(xy) the connectivity can only improve while $v \to x$ preserves the isomorphism type of the graph. Finally for `sub`$(xy; v)$ the property is easily seen directly. □ □

Note that if G is a 2-connected graph and there is a sequence of 2-admissible transformations which carries G into a spanning subgraph of some graph H, then H is also 2-connected.

Proposition 4. *Let* G *be a connected, locally* 2*-connected graph other than* K_3 *and let* $v \in V(G)$ *be a vertex with* $\deg_G(v) \leq 5$*. Then there exists a connected, locally* 2*-connected graph* H *such that*

$$V(H) = V(G) \setminus \{v\}, \quad E(G - v) \subseteq E(H) \quad \textit{and} \quad \beta_2(H) \leq \beta_2(G) \ .$$

Note that the last two conditions imply a double inequality

$$0 \leq |E(H)| - |E(G - v)| \leq \deg_G(v) - 3 \ . \tag{3}$$

Proof. For brevity let $n = |V(G)|$, $m = |E(G)|$ and $d = \deg_G(v)$. If G is connected, locally 2-connected and contains a vertex of degree at most 2 then it is easy to conclude that $G = K_3$, contradicting our assumption. Hence, $3 \leq d \leq 5$. There are 14 possible isomorphism types that $\mathrm{lk}_G v$ can have. We will show that in each of those 14 cases,

> one can add at most $d - 3$ edges between the vertices of $N_G(v)$, so that if G' is the graph thus obtained and $H = G' - v$, then H is connected and locally 2-connected. $(\star)$

This will complete the proof, since H has $n-1$ vertices and H has at most $m-d+(d-3)=m-3$ edges; hence

$$\beta_2(G)=m-3n=(m-3)-3(n-1)\geq\beta_2(H) \ .$$

It is obvious that H satisfies the other required properties.

We proceed to describe how to perform the operation ($\star$). Note that if $u\in V(G)\setminus N_G[v]$ then $\mathrm{lk}_G u$ is a spanning subgraph of $\mathrm{lk}_H u$, hence $\mathrm{lk}_H u$ is 2-connected regardless of which edges are actually added to $N_G(v)$. Moreover, H is connected by Lemma 2 applied to G' with $k=1$. The only non-trivial fact to be checked in order to prove ($\star$) is that $\mathrm{lk}_H u$ is 2-connected for vertices $u\in N_G(v)$. For this, we will verify the following refinement of ($\star$):

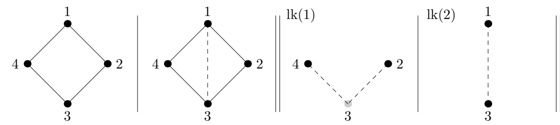
> in ($\star$), one can additionally ensure that for all $u\in N_G(v)$ the link $\mathrm{lk}_H u$ contains a spanning subgraph that is obtained from $\mathrm{lk}_G u$ by a sequence of 2-admissible operations. Moreover, those operations involve only the vertices of $N_G[v]$. ($\star\star$)

This proves ($\star$), since all graphs $\mathrm{lk}_G u$ are 2-connected. We will now check ($\star\star$) directly for all the possible isomorphism types of $\mathrm{lk}_G v$.

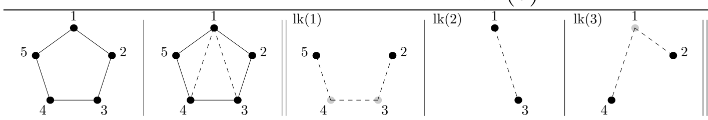
If $\mathrm{lk}_G v$ is one of K_3, K_4, K_4-e, K_5, K_5-e, K_5-2e or K_5-P_3, then one can add at most $d-3$ edges between the vertices of $N_G(v)$ making $H[N_G(v)]$ the complete graph K_d. Then, for every vertex $u\in N_G(v)$ we can pass from $\mathrm{lk}_G u$ to a spanning subgraph of $\mathrm{lk}_H u$ using the following operations: adding edges, adding new vertices of degree at least 2 and removing the vertex v whose link is a clique of size $d\geq 3$. All these operations are 2-admissible, so the claim is proved.

For the remaining 7 possibilities of $\mathrm{lk}_G v$ the proof is continued in Table 1. We read it as follows. The first column shows $\mathrm{lk}_G v$. The second column shows additionally the new (dashed) edges which will be added to $N_G(v)$, as required in ($\star\star$). In other words, the second column is $\mathrm{lk}_{G'} v$. The remaining columns depict the graphs $(\mathrm{lk}_H u)[N_G(v)]$ for $u\in N_G(v)$, ignoring the cases which differ just by a symmetry of the configuration. In each case the shaded/dashed elements represent vertices and edges which did not exist in $\mathrm{lk}_G u$ but appear in $\mathrm{lk}_H u$. The only element from $\mathrm{lk}_G u$ missing in $\mathrm{lk}_H u$ is the vertex v. Below each figure there is a sequence of 2-admissible operations which transforms $(\mathrm{lk}_G u)[N_G[v]]$ into $(\mathrm{lk}_H u)[N_G(v)]$, and therefore transforms $\mathrm{lk}_G u$ into a spanning subgraph of $\mathrm{lk}_H u$.

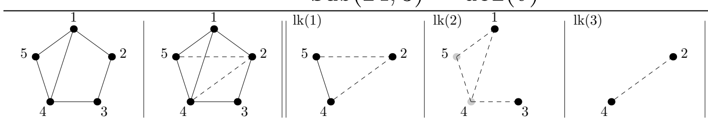
A direct verification of the entries in the table completes the proof of ($\star\star$) and hence of the proposition. □ □



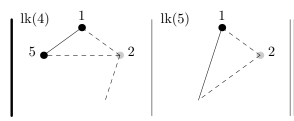
$v \to 3$	add(13) del(v)	



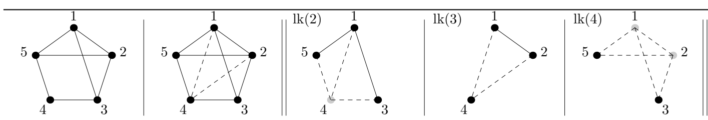
$v \to 4$ sub$(24;3)$	add(13) del(v)	$v \to 1$



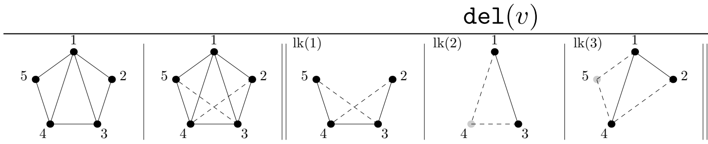
add(24) add(25) del(v)	$v \to 4$ add$(5;1,4)$	add(24) del(v)



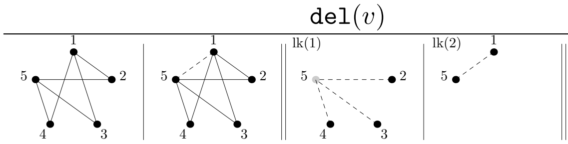
$v \to 2$	$v \to 2$	



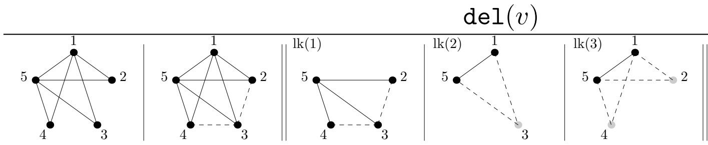
$v \to 4$	add(14) add(24) del(v)	$v \to 1$ add$(2;1,3,5)$



add(35) add(24) del(v)	$v \to 4$	add(24) add$(5;1,4)$ del(v)



$v \to 5$	add(15) del(v)	



add(23) add(34) del(v)	$v \to 3$	$v \to 2$ add$(4;1,5)$

Table 1: The essential part of the proof of Proposition 4.

Theorem 5. *Every connected, locally 2-connected graph G is M_3-rigid. As a consequence, $\beta_2(G) \geq -6$.*

Proof. Let G be a connected, locally 2-connected graph. Let $W \subseteq V(G)$ with $|V(G) \setminus W| \geq 3$. We wish to show that $|E(G,W)| \geq 3|W|$. If $G = K_3$ then the inequality holds, since we must necessarily have $W = \emptyset$. Now we proceed by induction on $|V(G)|$.

If every vertex $v \in W$ satisfies $\deg_G(v) \geq 6$, then we are done, since

$$|E(G,W)| \geq \frac{1}{2}\sum_{v\in W} \deg_G(v) \geq 3|W| \ .$$

If $v \in W$ is a vertex with $\deg_G(v) \leq 5$ then consider the graph H provided by Proposition 4. It is connected, locally 2-connected and satisfies $V(H) = V(G) \setminus \{v\}$. Since $v \in W$, we have

$$|V(H) \setminus (W \setminus \{v\})| = |V(G) \setminus W| \geq 3,$$

so the inductive assumption applies to the pair $(H, W \setminus \{v\})$, and we have

$$|E(H, W \setminus \{v\})| \geq 3(|W| - 1) \ .$$

Combining this with (3) we obtain

$$\begin{aligned} |E(G,W)| &= \deg_G(v) + |E(G - v, W \setminus \{v\})| \\ &\geq \deg_G(v) + |E(H, W \setminus \{v\})| - (\deg_G(v) - 3) \\ &\geq 3(|W| - 1) + 3 = 3|W|, \end{aligned}$$

which proves the M_3-rigidity of G and completes the inductive step.

To prove the second assertion, note that in a locally 2-connected graph G the link of every vertex is, in particular, a connected graph with at least two vertices. This means that every vertex of G belongs to a triangle, and the inequality $\beta_2(G) \geq -6$ follows from (2). □ □

Remark. Theorem 5 is not true if "locally 2-connected" is replaced with "locally 2-edge-connected". The smallest example is the graph $K_6 - C_4$. In this graph the link of each vertex is either K_3 or $K_5 - C_4$, the latter being the only 2-edge-connected, but not 2-connected graph on at most 5 vertices.

We also have a more general construction. The graph $K_6 - C_4$ has two edges e_1 and e_2 which connect pairs of vertices of degree 3. Let G_n denote the graph constructed from n copies of $K_6 - C_4$ arranged in a cyclic fashion and glued so that the edge e_2 of the i-th copy is identified with the edge e_1 of the $(i+1)$-th copy (indices modulo n). In the graph G_n the link of every vertex is $K_5 - C_4$, hence G_n is a sequence of locally 2-edge-connected graphs with increasing $|V(G_n)|$ and with $|E(G_n)| = 2.5 \cdot |V(G_n)|$.

4 Remarks and conjectures

Local 1-connectivity. Using the same method one can show that a connected, locally 1-connected graph G is M_2-rigid. This implies, and strengthens, the lower bound $\beta_1(G) \geq -3$ for such graphs [12]. We leave this as an exercise for the reader.

Local 2-connectivity. A connected graph G which is locally 2-connected is (globally) 3-connected [3]. By Tutte's theorem either $G = K_3$ or G has an edge e such that G/e is 3-connected, where G/e denotes the graph obtained by contraction of e. That suggests the next conjecture.

Conjecture. Any locally 2-connected graph G other than K_3 has an edge e such that G/e is locally 2-connected.

Note that an analogous result for locally 1-connected graphs was shown by Kriesell [6].

Local 3-connectivity. A computational search through small examples suggests that the following is still true.

Conjecture. A connected, locally 3-connected graph G is M_4-rigid. Moreover, we have $\beta_3(G) \geq -10$.

However, we were unable to prove an analogue of Proposition 4 for this case. Note that Conjecture 4 *implies* Theorem 5. This can be seen by taking a connected, locally 2-connected graph G and applying Conjecture 4 to the graph $G \oplus 1$ (see below), which is connected and locally 3-connected.

Higher local connectivity. We now consider locally 4-connected graphs. To this end, recall that for $k = 1, 2$ it follows from Theorem 5 and the work of Zelinka [12] that any connected, locally k-connected graph G satisfies the lower bound $|E(G)| \geq (k+1)|V(G)| - \binom{k+2}{2}$, which coincides with the bound for locally k-trees [2] and with the bound (2) for $d = k+1$. Also recall that for $k = 3$ we conjecture (Conjecture 4) this lower bound to hold.

If $k \geq 4$, this need not be the case. Consider the following examples, where $G \oplus H$ is the graph obtained from the disjoint union of G and H by adding all the edges between $V(G)$ and $V(H)$. A single natural number h stands for the graph with h vertices and no edges.

Example 6. For any $n \geq 4$ and $k \geq 2$ let

$$G_{k,n} = \begin{cases} 2^{\oplus(k/2)} \oplus C_n, & \text{when } k \text{ is even,} \\ 1 \oplus G_{k-1,n}, & \text{when } k \text{ is odd.} \end{cases}$$

The graph $G_{k,n}$ is connected and locally k-connected and one easily computes

$$|E(G_{k,n})| = (k+1)|V(G_{k,n})| - \frac{1}{2}(k^2 + 4k - [k \bmod 2]) \ .$$

For $k \geq 4$ it holds $\frac{1}{2}(k^2+4k-[k \bmod 2]) > \binom{k+2}{2}$. Moreover, since $|V(G_{k,n})|$ is an increasing and unbounded function of n, this shows that for $k \geq 4$ the linear term $(k+1)|V|$ in Conjecture 1 is best possible. □

Note that for $k \geq 2$, we have inequalities

$$F(k) \geq \frac{1}{2}(k^2 + 4k - [k \bmod 2]) \geq \binom{k+2}{2} \ . \tag{4}$$

For $k = 2$, we have equality throughout in (4) by our main result (Theorem 5). Conjecture 4 asserts that we also have equality throughout in (4) for $k = 3$. For $k = 4$, inequalities (4) become $F(4) \geq 16 > 15$, and we conjecture that $F(4) = 16$.

Conjecture. For any connected and locally 4-connected graph G,

$$|E(G)| \geq 5|V(G)| - 16 \ .$$

The motivation for the above conjecture is the following. As we explained in the introduction, Theorem 5 can be seen as a relaxation of the restrictive condition "surface triangulation" into the more robust "locally 2-connected". Here we are pursuing the same analogy, where the more restrictive condition we are trying to relax is "flag triangulation of a 3-manifold" [5]. To be more precise, suppose that G is the 1-skeleton of a flag triangulation of the 3-sphere S^3. Then G is connected and locally 4-connected, by a result of Athanasiadis [1]. It is a deep result of Davis and Okun [4] that Conjecture 4 holds for such graphs.

It is difficult to speculate what the best constant $F(k)$ in Conjecture 1 could be for higher k (assuming it is finite). For example, there is a connected, locally 6-connected graph G for which $\beta_6(G) = -32$.[2] It means that $F(6) \geq 32$, while the last two entries in (4) are 30 and 28, respectively. It follows that the construction in Example 6 is not optimal for $k \geq 6$.

[2] It is `O~~em]uj[vmsZTUrfFwN~` in the notation of the graph package `nauty` [7].

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